

BALLISTIC RESEARCH LABORATORIES



REPORT No. 784

Spatial Triangulation by Means of Photogrammetry

PROPERTY OF U.S. ARMY
STINFO BRANCH
BRL, APG, MD. 21005

H. SCHMID

ABERDEEN PROVING GROUND, MARYLAND

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Project No. TB3-0838 of the Research and
Development Division, Ordnance Corps

ABERDEEN PROVING GROUND, MARYLAND

Table of Contents

	<u>Page</u>
Abstract - - -	3
I. Introduction - -	5
II. Photogrammetry by Intersection -	6
III. Orientation of a Photogrammetric Camera -	11
1. The photogrammetric camera and the perspective properties of the photograph -	11
2. The mathematical analysis of the orientation problem. -	13
3. The rigorous least squares adjustment. -	31
IV. Mathematical Analysis of the Spatial Triangulation Problem -	52
V. Accuracy of the Method -	57
VI. Numerical Examples - -	59
VII. Summary - -	83
Acknowledgements - - -	84
Distribution List - - -	85

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HSchmid/els
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ABSTRACT

A method of precision spatial triangulation based on the principles of ground photogrammetry is outlined. The geometry and the least squares adjustment for the orientation of an individual photogrammetric camera are derived and the mathematical analysis for the spatial triangulation by means of intersection photogrammetry is given. Numerical examples are added for the major steps of the developed method.

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I. INTRODUCTION

Geodetic measuring methods are characterized by the measuring of angles and distances. The corresponding problems in photogrammetry are the determination of angles and distances from photographs. The problem may be considered in two ways, which are basically different in their geometrical approach. Either these quantities may be determined analytically from the coordinates measured, or they may be derived projectively by reproducing the bundles of rays, for each exposure of the system of cameras. Only in special cases does there exist a simple relation between distances measured on a photograph and on the space object respectively. In general, however, photogrammetry deals only with the determination of angles and this report will deal with such measurements exclusively. The general problem of photogrammetric measurements may, therefore, be defined as the triangulation of spatial positions from angle measurements. As in geodesy, one uses in photogrammetry either the intersection or resection methods.

The purpose of this report is to outline a method of precision spatial triangulation for aerial targets recorded on photographs, taken at ground stations. Therefore, we are dealing with a problem of ground photogrammetry. Basic conditions of conventional ground photogrammetry, as distinguished from aerial photogrammetry, are:

1. The camera stations are on the earth.
2. The camera stations do not change their position with time, so that it is justified to separate the geodetic position measurements for the camera stations from the photogrammetric measuring procedure.

However, one of the most important characteristics of conventional ground photogrammetry should not be applied in our case, i.e., the assumption that the relation between the space position of the plate and the local plumbline direction, as obtained by levels, can be used as a parameter in combining several stations for the purpose of triangulation. The physical reasons for this limitation are explained in Chapter II of this report. Consequently, not all the unknown elements of orientation can be determined separately as is done conventionally. The elements of the exterior orientation defined, e. g., by the azimuth and tilt angles of the optical axis and by the swing angle of the plate, must be determined simultaneously. The nature of the rigorous least squares solution for the plate orientation, given in Chapter III/3, makes it necessary to determine these three quantities in connection with the computation of the three elements of the interior orientation, denoted by the principal distance and the two plate coordinates of the principal point. These complications call for modifying the evaluation procedure from the conventional ground methods to those of aerial photogrammetry. The latter are characterized by the fact that the determination of all parameters of orientation must be made simultaneously because of the changing position of the camera.

The final triangulation of the recorded aerial targets is obtained from the projective relations which exist between pairs and triplets of photographs made from different stations. Thus the spatial triangulation is carried out by intersections or resections. In the outlined method, each station is treated as an independent unit except for the final step when corresponding rays are combined in a spatial triangulation. The present report is, therefore, limited to a treatment of the problem based on the principles of "photogrammetry by intersection". Consequently, the photogrammetric measuring procedure is applied at each measuring station independently. The results for each recorded target point are expressed in parameters which are related only to the particular measuring station, e. g., by two position angles.

This method is not the only photogrammetric method for triangulating the spatial position of a point. As a matter of fact, a more elegant and more economic method may be used. This is based on the projective relations between photographs taken from different locations, which may not only serve to triangulate recorded points, but present the possibility to determine the relative orientation of these photographs exclusively from their projective properties. In connection with such a method, it is possible to measure two plates simultaneously with stereoscopic reading devices. These methods will lead to a mechanical-optical solution of our problem with the aid of a high-precision stereo plotting machine. This is essentially a three-dimensional stereoscopic comparator by which the spatial triangulation problem is simulated.

Although this paper deals only with the method of photogrammetry by intersection, the mathematical analysis of the problem of plate orientation is presented in a form which may be suitable for future stereophotogrammetric measuring methods.

II. PHOTOGRAMMETRY BY INTERSECTION

The spatial triangulation of a recorded point from photographs taken at two ground stations may be computed in ground photogrammetry by the intersection method. (Fig. 1) The horizontal directions and the elevation angles, necessary for the intersection, are obtained if the principal distances d are known, by measuring the plate coordinates (\bar{x} and \bar{y}). The directions of the camera axes against the base line or any other reference datum are obtained either directly by readings of horizontal and vertical circles, or indirectly with the aid of additional reference points, whose space coordinates must be known. The use of dials so far has not proved to be sufficiently accurate to eliminate the need for control points. Besides the unavoidable systematic instrumental errors, such a method is affected by deflections of the vertical at the stations, since the setting of the phototheodolite (photogrammetric camera plus theodolite) is obtained with reference to the vertical as indicated by levels. Precision photogrammetry should, therefore, use the data of exterior orientation obtained

Triangulation by Intersection

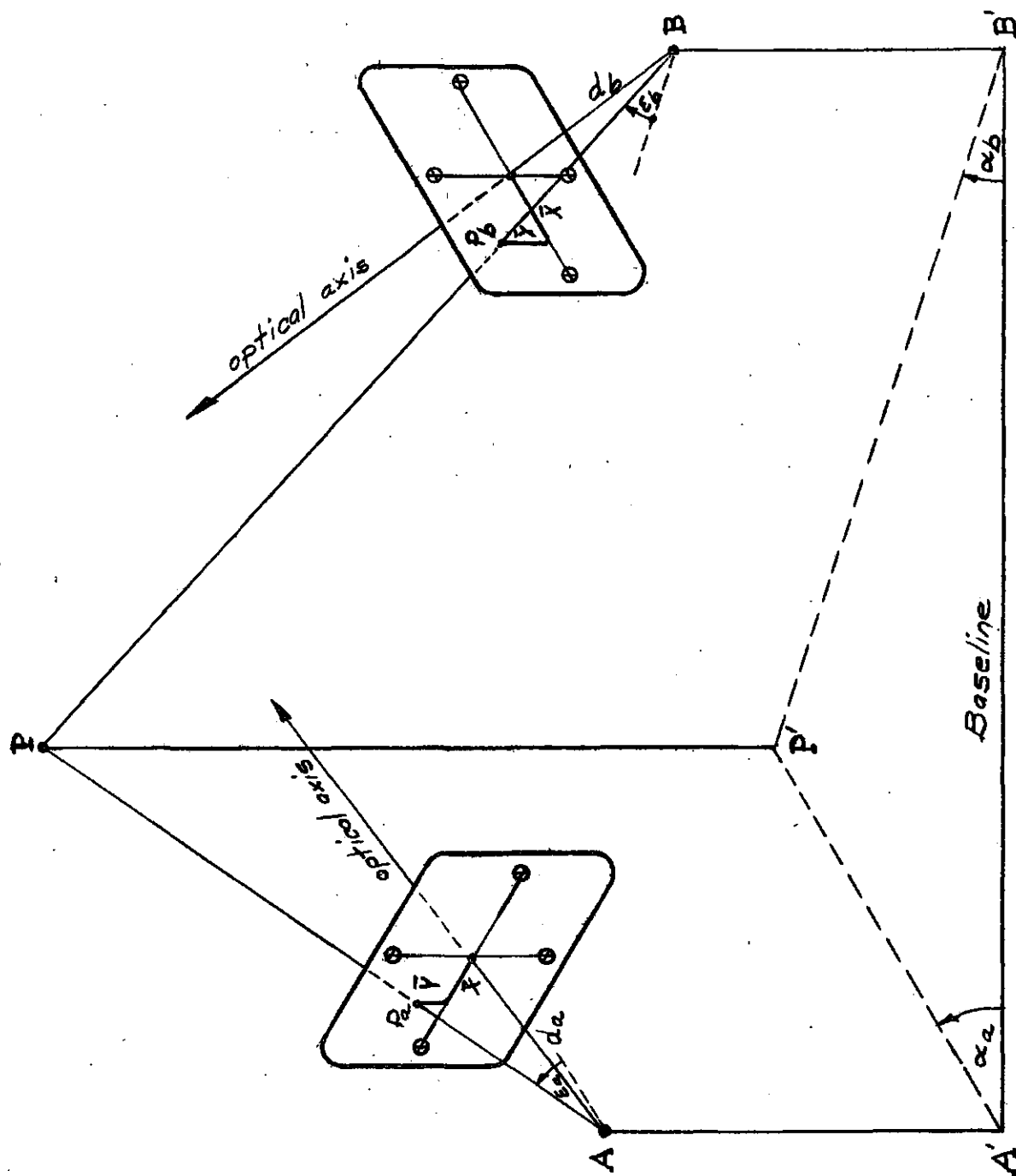


Figure 1

from dial readings with or without correction for plumbline anomalies as approximation values only and evaluate the final orientation of the plate with the aid of recorded control points. It is evident that the coordinates of such reference points must be recorded in a way such that they can be accurately identified and measured. It should be mentioned here that the use of control points in geodetic practice is considered essential, especially in order to eliminate the influence of systematic instrumental errors on the final plate orientation. The fact that it is possible to compute from recorded control points corrections to the orientation elements of the plate, shows that the elements of orientation may be obtained exclusively from measurements of such control points. Hence, dials and levels are not basic elements of photogrammetric instrumentation design. They may be useful to increase the computing economy but it should be understood that the parameters obtained from such auxiliaries should be considered only as approximations. Consequently, a theory for a photogrammetric measuring method for precision triangulation must be established for a completely independent photogrammetric camera. The so-called "Ballistic Camera", which is used, e. g., to record ballistic data of full-scale missiles in flight, represents this type of photogrammetric camera.

Summarizing, the interior and exterior orientation of a camera may be computed in a given coordinate system from recorded control points if the coordinates of such points are given within the same reference datum. Hence, the direction to an additional recorded target point may be obtained from the computed plate orientation and the measured plate coordinates of the point. The result may be expressed by two position angles, e. g., by an azimuth and elevation angle. If such results from two or more stations are combined, the space position of the target point may be found by intersection.

The determination of space positions of missiles from photographs taken at several ground stations may be considered as a similar problem. However, this problem is different in that the camera axes are directed into space so that in general no terrestrial reference points will be within the angle of view of the camera lens or at least they will be recorded only on the edge of the plate. In such a case, it is possible to use stars as control points. However, this procedure will give rise to some difficulty which will be considered next. We have seen that the control points must be known in the same coordinate system in which the final orientation of the plate and the final measuring data are required. This makes it necessary to establish the relation between the right ascension and declination coordinates of the stars and the local earth fixed coordinate system of the measuring stations. It is well known that there are formulas which allow us to express the azimuth and elevation angles of a star as functions of right ascension, declination, sidereal time and geographic latitude of the station. The right ascension and declination values as well as the time measurements may be obtained with sufficient accuracy. Hence, the problem is to determine the geographic coordinates of the stations. The determination of the latitude and longitude of the stations may be obtained by astronomical observations. These

results, however, are not useful, due to the fact that the measurements are affected by vertical deflections, caused mainly by mass irregularities in the crust of the earth. The same problem exists in the astronomical determination of the azimuth of a baseline. Another source for obtaining the geographic coordinates are the triangulation results. However, triangulation systems must be considered as local constructions distorted by unknown amounts. Especially poor is the orientation of extended triangulations. In addition the assumption is not justified that the primary orientation of the reference ellipsoid is so good that the relation between the celestial and the earth fixed system can be obtained accurately enough to serve as a basic parameter for a precision triangulation by two or more photogrammetric stations.

In the following, a method is developed in which this difficulty will be eliminated. It is obvious that the orientation of the plate may be obtained in the right ascension - declination system from star measurements. With the measured plate coordinates for any recorded aerial target point, the position of such a point in the sky may be expressed by right ascension and declination angles, thus treating the recorded point as an artificial star. Since the stars are at essentially infinite distance, a given portion of the sky will show an identical pattern independent of the location of the observer. However, the apparent place of the target point in the sky will change with the position of the observer. Therefore, the target point will have a different set of right ascension and declination values for each measuring station. Combining the results of two stations, it is possible to determine the spatial angle between two such directions (angle σ in Fig. 2). If a third measuring station is combined with each of the two other stations, three such spatial angles will be determined. By combining the angles with the slant distances between the measuring stations, which may be computed from local geodetic data, a pyramid is formed. We now have a typical problem of photogrammetry, namely, that of determining the coordinates of a point in space from three position angles with reference to three known points, i.e., the well-known problem of resection in space. With this step the triangulation problem is solved without resorting to the critical values of geographic coordinates. The result is obtained by making the transition from the celestial to the earth-fixed system at that stage in the computing procedure when a quantity is obtained whose magnitude is independent of the coordinate system used. The spatial angles between the determining rays of the target point are such independent parameters. It is noted, that if the exposures are made synchronously at all measuring stations, the time does not enter into the solution, with the exception that an approximate sidereal time is needed to compute the refraction.

* Compare BRL Report No. 746, H. Schmid, "Principles of Physical Geodesy for the Establishment of a Long-Range Weapons Test Range."

TRIANGULATION BY RESECTION

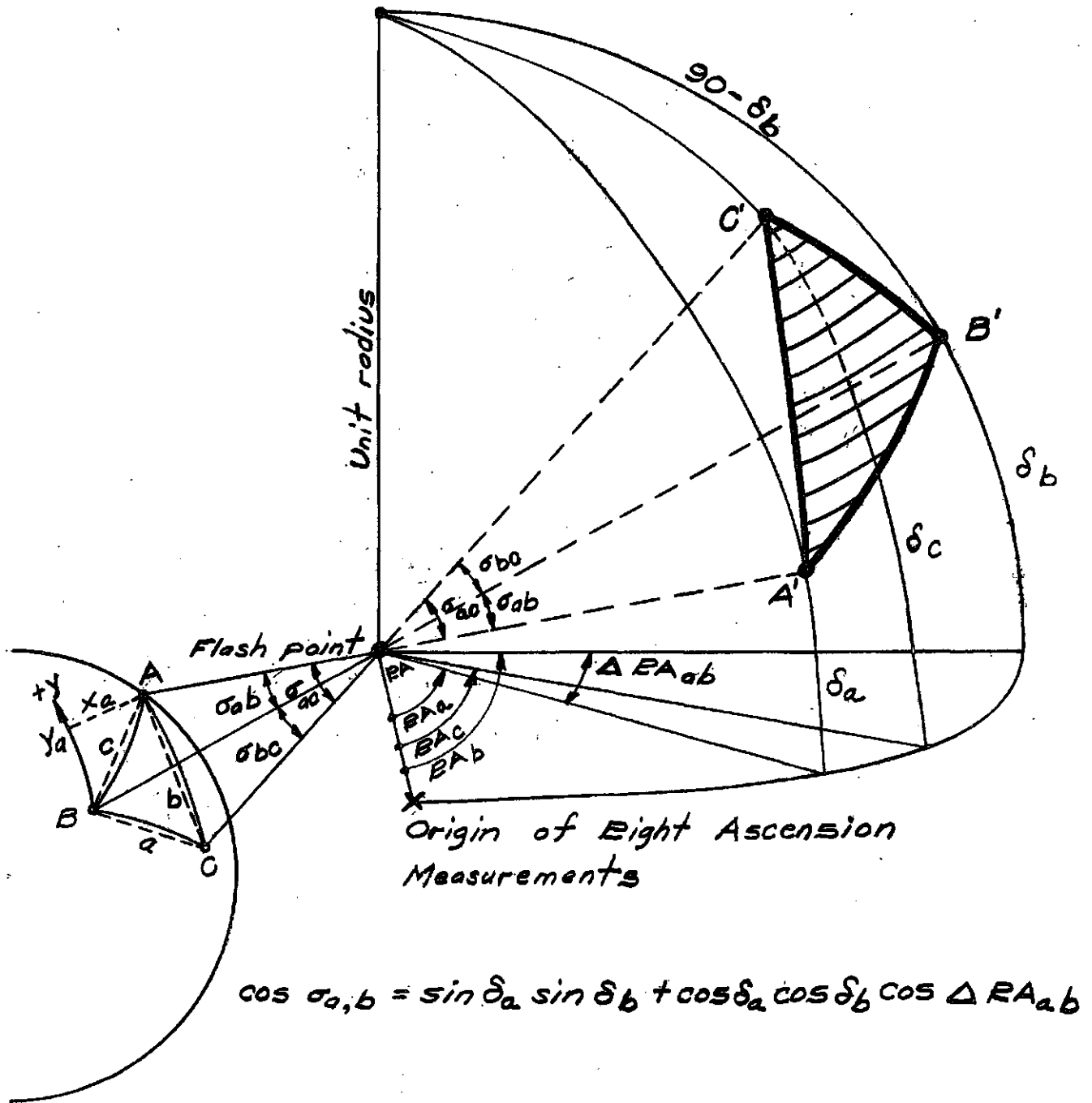


Figure 2

If the star exposures at the different stations are made at different times, this time difference only must be known accurately. This measurement will enter into the solution as a corresponding correction to the right ascension values.

The conclusion is that for the establishment of a precision photogrammetric measuring system, using stars as control points, three stations are sufficient and necessary in order to obtain a rigorous solution for a spatial triangulation. A triangulation by intersection from any two stations may be considered as adequate only if a systematic orientation error of the base-triangle has been determined from the results of a former rigorous three-station solution. Thus, a correction to the azimuth of the local triangulation system would be determined. This procedure will be referred to in this report as the "calibration of a photogrammetric base line".

III. ORIENTATION OF A PHOTOGRAMMETRIC CAMERA

1. The photogrammetric camera and the perspective properties of the photograph.

In a previous chapter, it has been shown that a precision photogrammetric triangulation method should be based on measurements obtained from an independent photogrammetric camera, i.e., measurements not dependent upon azimuth and elevation dial readings. Mathematically, we have to consider only the optical system as the center of projection and the photographic plate as part of a plane cutting a bundle of perspective rays. Each perspective may be explained as a figure obtained by cutting a bundle of perspective rays with a plane. Consequently, each photograph represents a perspective figure. We distinguish between the diapositive if the plane is situated between the object points and the center of projection, and the negative, if the center of projection is located between the object points and the image plane.

The relative situation of the individual perspective rays of such a bundle is uniquely determined, and it may be congruently reconstructed if the distance of the perspective center (in practice, the rear nodal point) from the image plane, i. e., the principal distance d , and the location of the principal point on the image plane are known. The "principal point" is the intersection of a line perpendicular to the plate through the perspective center. This line is called the photograph perpendicular. The location of the principal point on the plate may be given by two rectangular coordinates in any plate coordinate system represented by certain fiducial marks, and be denoted by Δx and Δy . (Fig. 3) The principal distance and the coordinates of the principal point are the three elements of interior orientation. A photograph of which the interior orientation is known, is called a photogram.

Elements of
interior and exterior
orientation

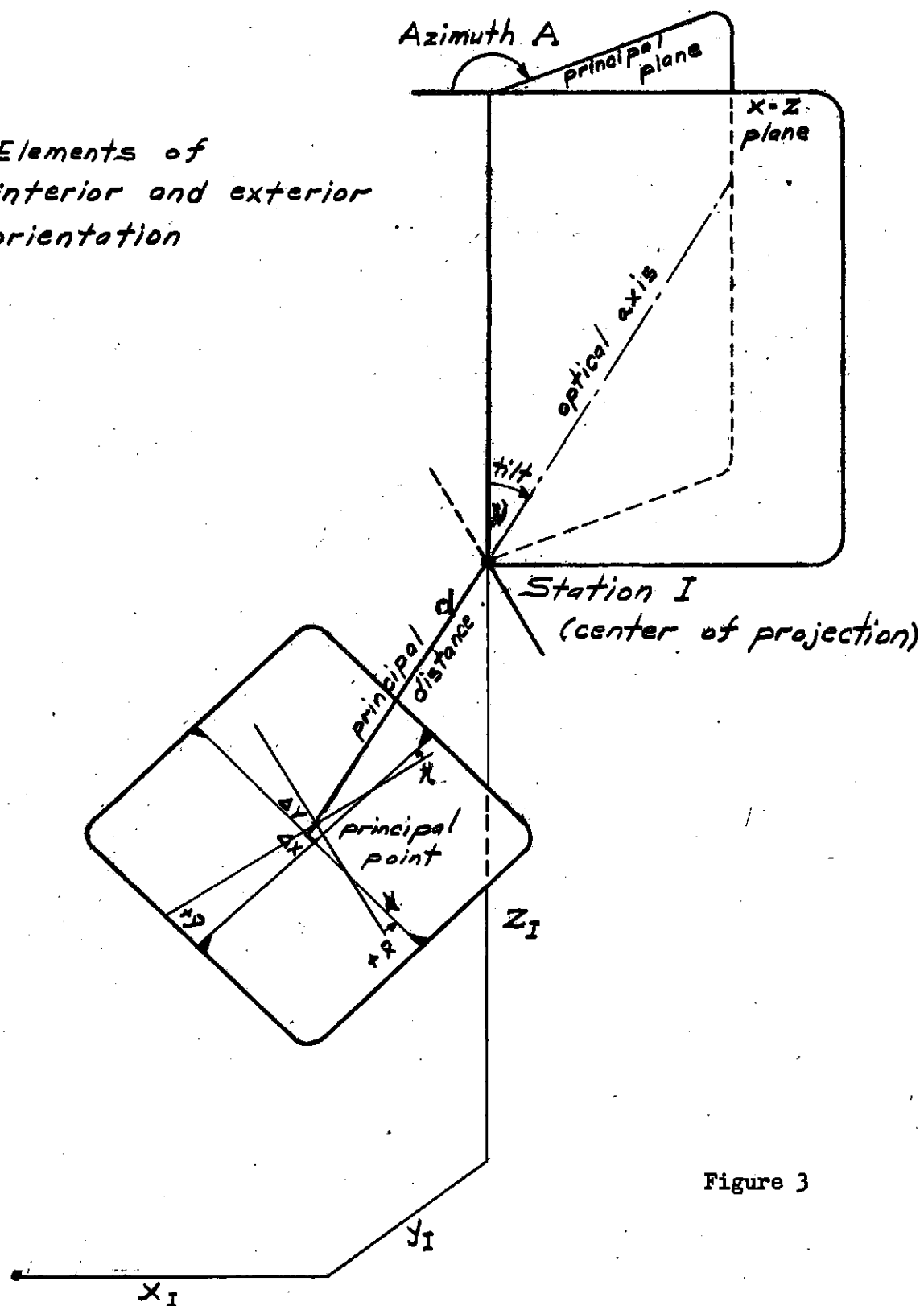


Figure 3

Thus, the position of the center of projection is determined by three perspective rays. The same three rays determine the reciprocal position of the center of projection with respect to the corresponding point-objects. The three space coordinates of the center of projection, (X_I, Y_I, Z_I) , the spatial direction of the plate perpendicular (optical axis) expressed, e.g., by two position angles (A and ψ) and the swing angle (κ) of any reference line on the plate, represent the elements of exterior orientation. (Fig. 3) Hence, the exterior orientation is determined by six parameters.

2. Mathematical Analysis of the Orientation Problem

In the preceding chapter, we have seen that the interior orientation of a camera calls for the determination of three parameters and that the exterior orientation requires six elements. In our case the space position of the center of projection may be determined separately from geodetic measurements, thus reducing the number of unknown elements for the exterior orientation to three. Hence, the absolute orientation of a plate in our case calls for the determination of six elements, three elements of interior and another three elements of exterior orientation. These six parameters of the absolute orientation are:

- | | | |
|---|---|--|
| (1) Principal distance - d | } | The three elements
of interior
orientation |
| (2) x-coordinate of the principal point - Δx | | |
| (3) y-coordinate of the principal point - Δy | | |
| (4) the azimuth angle of the plate perpendicular - A | } | The three elements
of exterior
orientation |
| (5) the tilt angle of the plate perpendicular - ψ | | |
| (6) the swing angle of the plate coordinate system - κ | | |

We have seen that the orientation of the plate must be determined from recorded stars as control points. Therefore, the elements of exterior orientation are determined with reference to the right ascension-declination system.

The right ascension and declination coordinates are rectangular spherical coordinates and may be represented on a unit sphere. In order to relate these spherical reference coordinates to the plane plate coordinates, a transformation of one of the two systems is necessary. With regard to future theoretical work, it was decided to transform the spherical star coordinates into plane coordinates by projecting the stars in a plane tangent to the unit sphere. The plane coordinates of the stars are obtained according to the principle of central projection with the point of projection in the center of the sphere (Fig. 4). If the plane is tangent at the celestial pole and the coordinate system is oriented in such a way that the ξ -axis represents the celestial meridian through the origin of the right ascension measurements and the η -axis is perpendicular to the

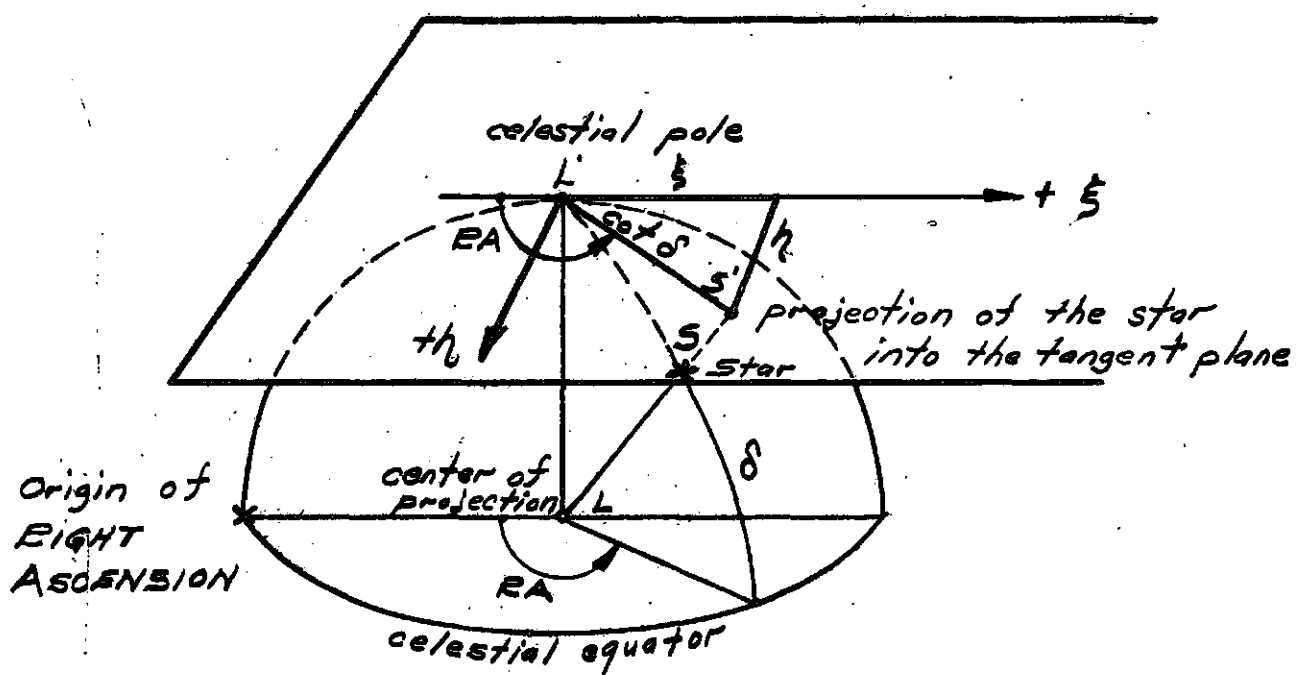


Figure 4

ξ -axis, the plane (standard) coordinates ξ and η for a star may be expressed as functions of right ascension (RA) and declination (δ). The formulas may be read directly from Fig. 4. They are:

$$\begin{aligned} +\xi &= -\cot \delta_r \cos RA_r \\ +\eta &= +\cot \delta_r \sin RA_r \end{aligned} \quad (1)$$

The subscript "r" indicates that the value has been corrected for refraction. The astronomical refraction for the stars may be computed as a function of zenith distance with the conventional formulas used in astronomical and geodetic practice. The effect of refraction on the right ascension and declination of a star may be obtained with well-known formulas derived, e. g., in "Textbook on Spherical Astronomy" by W. M. Smart, or in "Elements of Practical Astronomy" by W. W. Campbell.

Depending on the geographical position of the camera station, it may sometimes become necessary to choose a plane tangent to the celestial equator, preferably at the point of origin of right ascension. In addition, we will have need for a plane tangent at the zenith of the camera station. In such a case, the ξ -axis represents the meridian through the measuring station. The η -axis is perpendicular to the ξ -axis. The standard coordinates are in such a case:

$$\begin{aligned} +\xi \text{ (north)} &= -\tan z_r \cos A = \frac{\tan \delta_r - \tan \phi \cos t_r}{\cos t_r + \tan \phi \tan \delta_r} \\ +\eta \text{ (east)} &= -\tan z_r \sin A = \frac{-\sec \phi \sin t_r}{\cos t_r + \tan \phi \tan \delta_r} \end{aligned} \quad (2)$$

z = Zenith distance of the star

A = Azimuth of the star, counted clockwise from the south.

ϕ = Geographic latitude of the camera station

t = Hour angle - Sidereal time of exposure (θ) minus Right ascension

We have seen that a photograph may be considered as an exact central projection. (The necessary corrections to the measured plate coordinates due to distortion will be discussed later). Therefore, a star photograph represents a central projection of a certain portion of the sky. Similarly, the projection of the stars in a plane tangent to a unit sphere is an exact central projection. Both projections may be assumed to have the point of projection in common. Hence, the plate images denoted by the plate coordinates \bar{x} and \bar{y} and the corresponding star projections denoted by the standard coordinates ξ and η , may be considered as photographs taken from the same point. We must now determine the geometric relations which exist between two photographs taken from the same point. In accordance with the two different geometric approaches to the problem, there are two different mathematical analyses.

First we will consider the projection method, which is felt to be more intuitive, because the unknowns of the solution have precisely defined physical meanings.

Solution a)

In Fig. (5), L is the center of projection. The problem is to orient the plane containing the plate which is shown as a diapositive, in such a way that a bundle of rays originating from L and passing through the plate images (S) intersects a plane tangent to the unit sphere at the point L' in the corresponding star projections, (S'). We introduce the following coordinate systems:

ξ, η Standard coordinates. Plane rectangular coordinates of a star in the plane of projection. The origin is at the point L', in which the projection plane is tangent to the unit sphere. ξ represents the meridian on the unit sphere through the point of origin of the azimuth measurements.

\bar{x}, \bar{y} Primary plate coordinates. Plane, rectangular coordinates of an image point, related to a system whose origin (0) is the intersection of the \bar{x} and \bar{y} axes, established by fiducial marks.

x, y Oriented plate coordinates. Plane, rectangular coordinates of an image point. The y-axis is the line of intersection of the image plane with a plane perpendicular to the plane of projection which contains the plate perpendicular. The origin is the isocenter I. The x, y system is tilted against the \bar{x}, \bar{y} system by the swing angle K . (K counted counter-clockwise)

x', y' Plane rectangular coordinates of a star in the plane of projection. The system corresponds in its orientation to the x, y system. The origin is the corresponding isocenter I'.

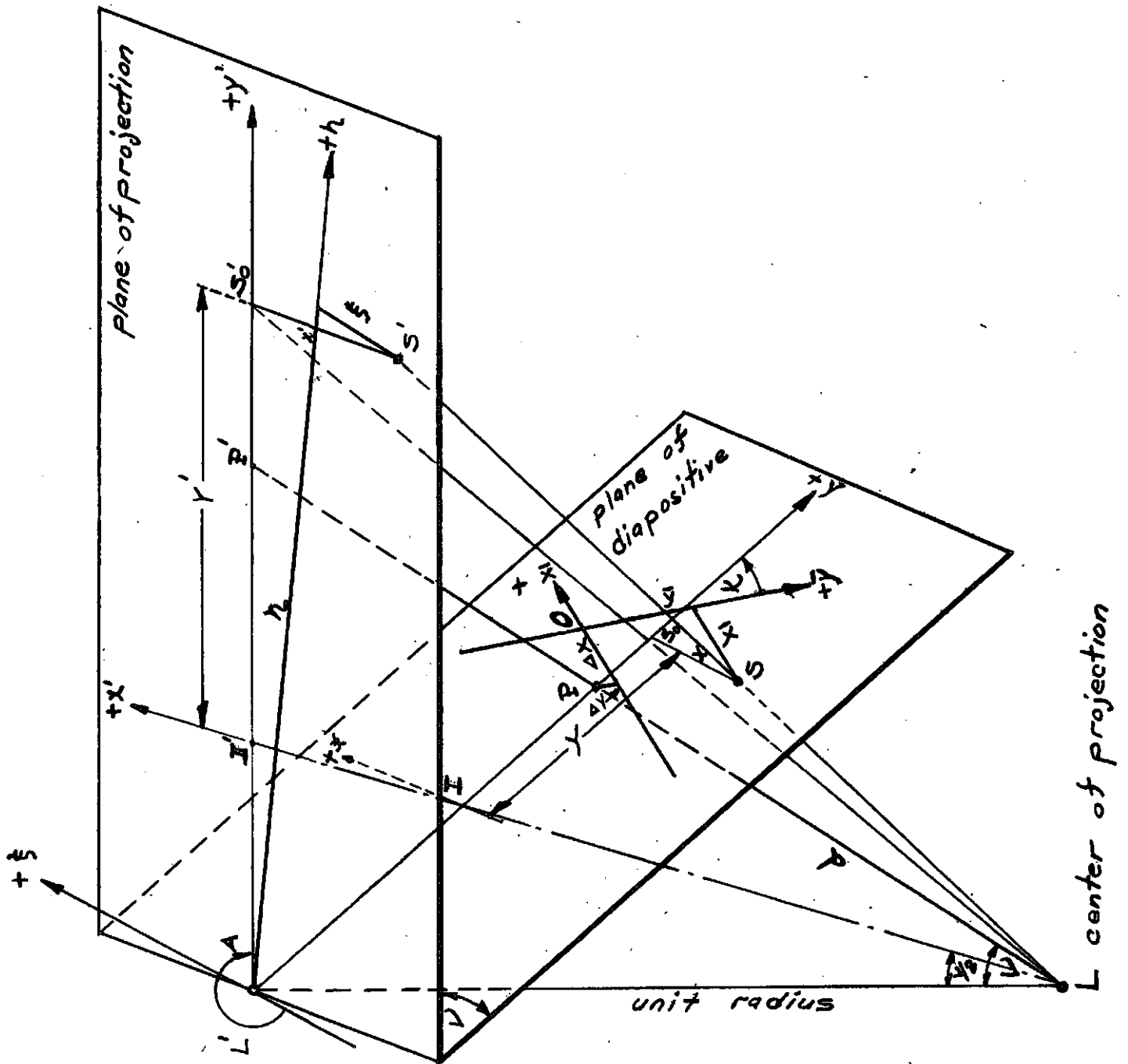


Figure 5

Figure 6

Δx and Δy denote the coordinates of the principal point (P) in the \bar{x} , \bar{y} system.

A = azimuth of the plate perpendicular

v = tilt of the plate perpendicular

d = principal distance

$$\begin{aligned} x &= (\bar{x} - \Delta x) \cos K - (\bar{y} - \Delta y) \sin K \\ y &= (\bar{y} - \Delta y) \cos K + (\bar{x} - \Delta x) \sin K + d \cdot \tan \frac{v}{2} \end{aligned} \quad (3)$$

From Figures 5 and 6, we obtain

$$HI = HL = \frac{d}{\sin v} \quad (4)$$

$$GI' = GL = \frac{h}{\sin v}$$

$\triangle GLS_o' \sim \triangle HS_o L$, therefore

$$\frac{GS_o'}{GL} = \frac{HL}{HS_o} \quad (5)$$

and

$$\frac{x'}{x} = \frac{LS_o'}{LS_o} = \frac{GS_o'}{LH} = \frac{GL}{HS_o} \quad (6)$$

Since $HS_o = LH - y$ and $GS_o' = GL + y'$, we have, from formulas 4, 5, 6,

$$y' = \frac{GL \cdot y}{HL - y} = \frac{hy}{d - y \sin v} \quad \text{and} \quad x' = \frac{GL \cdot x}{HL - y} = \frac{hx}{d - y \sin v} \quad (7)$$

Furthermore:

$$\xi = -x' \sin A - (y' + h \tan \frac{\nu}{2}) \cos A \quad (8)$$

$$\eta = -(y' + h \tan \frac{\nu}{2}) \sin A + x' \cos A$$

For our case $h = 1$ and therefore

$$y' = \frac{y}{d-y \sin \nu} \quad \text{and} \quad x' = \frac{x}{d-y \sin \nu} \quad (7')$$

$$\xi = -(y' + \tan \frac{\nu}{2}) \cos A - x' \sin A \quad (8')$$

$$\eta = -(y' + \tan \frac{\nu}{2}) \sin A + x' \cos A$$

From (7') and (8') we derive

$$\xi = \frac{+y \cos A + x \sin A}{-d + y \sin \nu} - \tan \frac{\nu}{2} \cos A \quad (9)$$

$$\eta = \frac{y \sin A - x \cos A}{-d + y \sin \nu} - \tan \frac{\nu}{2} \sin A$$

Substituting (3) into (9)

$$\xi = \frac{\{[(\bar{y}-\Delta y)\cos K + (\bar{x}-\Delta x)\sin K] \cos \nu + d \sin \nu\} \cos A + [(\bar{x}-\Delta x)\cos K - (\bar{y}-\Delta y)\sin K] \sin A}{+ [(\bar{y}-\Delta y)\cos K + (\bar{x}-\Delta x)\sin K] \sin \nu - d \cos \nu} \quad (10^a)$$

$$\eta = \frac{\{[(\bar{y}-\Delta y)\cos K + (\bar{x}-\Delta x)\sin K] \cos \nu + d \sin \nu\} \sin A - [(\bar{x}-\Delta x)\cos K - (\bar{y}-\Delta y)\sin K] \cos A}{+ [(\bar{y}-\Delta y)\cos K + (\bar{x}-\Delta x)\sin K] \sin \nu - d \cos \nu}$$

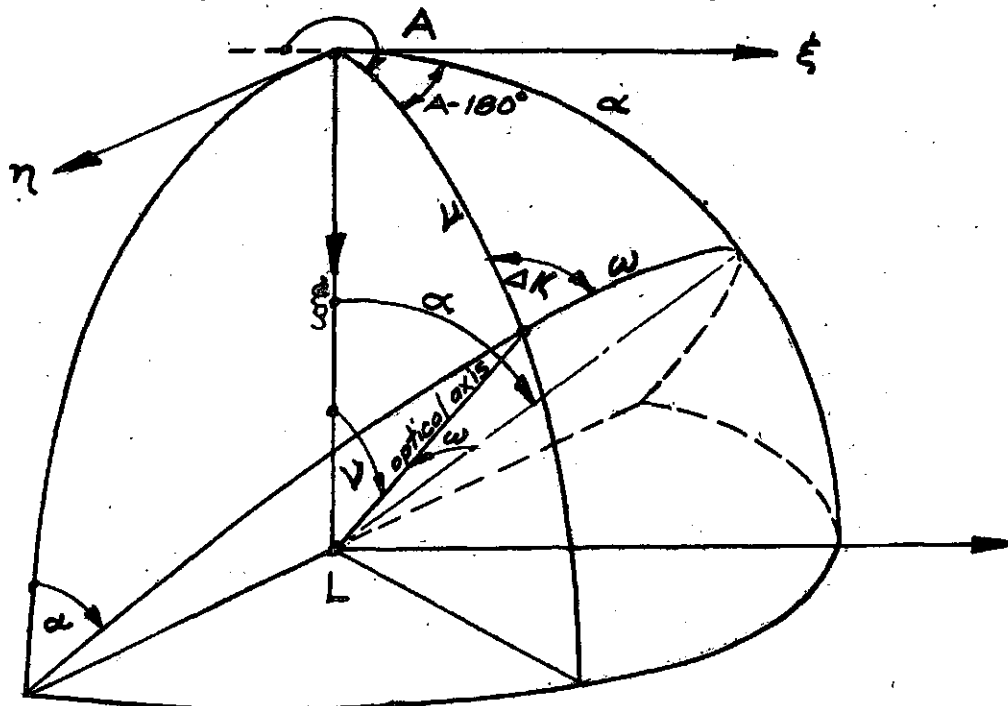
The formulas (10^a) are in agreement with the wellknown formulas Nos. 12a and 12b derived by O. v. Gruber.

The formulas (10^a) express the standard coordinates ξ and η in terms of the measured plate coordinates \bar{x} and \bar{y} in any rectangular plate coordinate system and in the six elements of orientation, namely, the three elements of interior orientation d , Δx , and Δy and the three elements of exterior orientation, A , ν , and K . These six parameters are the unknowns of the solution. Because each star gives rise to two equations, one for ξ and one for η , three stars are necessary and sufficient to obtain a unique solution. This result is in agreement with the previously stated principle that the center of projection is fixed with respect to the plate as well as object-points by any three perspective rays.

Solution b)

In the preceding paragraph, the spatial direction of the optical axis within the chosen reference system was expressed by two position angles, namely an azimuth angle (A) and a tilt angle (ν) (Fig. 5). Although these parameters are essential for the final triangulation problem, such an interpretation of the problem sometimes causes difficulties due to the fact that the concept of an azimuth angle does not exist anymore if the tilt angle ν becomes zero. In order to have formulas available which are useful in such a case, the spatial direction of the optical axis may be defined by the two tilt angles α and ω . (Fig. 7)

Figure 7



* Ferienkurs in Photogrammetrie by O. v. Gruber, Stuttgart-Verlag von Konrad Wittwer, 1930.

α, ξ -tilt, denotes the angle between the $\eta\xi$ plane and a plane parallel to the η -axis which contains the optical axis. ω, η -tilt, denotes the angle formed by the optical axis and its projection in the $\xi\xi$ -plane. The relations between the α, ω, κ angles and the A, v and (κ) angles respectively, are:

$$\begin{aligned}\sin \omega &= -\sin v \sin A & \cos v &= \cos \alpha \cos \omega \\ \tan \alpha &= -\tan v \cos A & \tan A &= \tan \omega \operatorname{cosec} \alpha \\ \cot \Delta \kappa &= \cos v \tan A & \kappa &= \Delta \kappa - (\kappa) \\ & & \text{where } (\kappa) &= \text{swing angle of method a.}\end{aligned}$$

The relations between the standard coordinates ξ and η and the measured plate coordinates \bar{x}, \bar{y} may be obtained by applying the formulas (5) twice. The result, obtained by a double projection, is as follows:

$$\begin{aligned}\xi &= \frac{[(\bar{y}-\Delta y)\sin\kappa + (\bar{x}-\Delta x)\cos\kappa]\cos\alpha + \{d\cos\omega - [(\bar{y}-\Delta y)\cos\kappa - (\bar{x}-\Delta x)\sin\kappa]\sin\omega\}\sin\alpha}{\{d\cos\omega - [(\bar{y}-\Delta y)\cos\kappa - (\bar{x}-\Delta x)\sin\kappa]\sin\omega\}\cos\alpha - [(\bar{x}-\Delta x)\cos\kappa + (\bar{y}-\Delta y)\sin\kappa]\sin\alpha} \\ \eta &= \frac{d\sin\omega + [(\bar{y}-\Delta y)\cos\kappa - (\bar{x}-\Delta x)\sin\kappa]\cos\omega}{\{d\cos\omega - [(\bar{y}-\Delta y)\cos\kappa - (\bar{x}-\Delta x)\sin\kappa]\sin\omega\}\cos\alpha - [(\bar{x}-\Delta x)\cos\kappa + (\bar{y}-\Delta y)\sin\kappa]\sin\alpha}\end{aligned}\quad (10^b)$$

These formulas agree with Gruber's formulas Nos. 13a and 13b*.

Solution c)

We will now consider the mathematical analysis of the orientation problem based on the principles of photogrammetry by analytical means. In Fig. (8), L is the center of projection. A bundle of rays is cut by two planes. One plane contains the plate (negative), the other plane represents a plane tangent to the unit sphere at point L' with the center at L. A star image in the plane containing the plate is denoted by S. The corresponding image in the plane of projection is S'. We again introduce the following coordinate systems:

ξ, η Standard coordinates. Plane rectangular coordinates of a star in the plane projection. The origin is at the point L', on which the plane is tangent to the unit sphere. ξ represents the meridian on the unit sphere through the point of origin of the azimuth measurements.

* See reference note on preceding page.

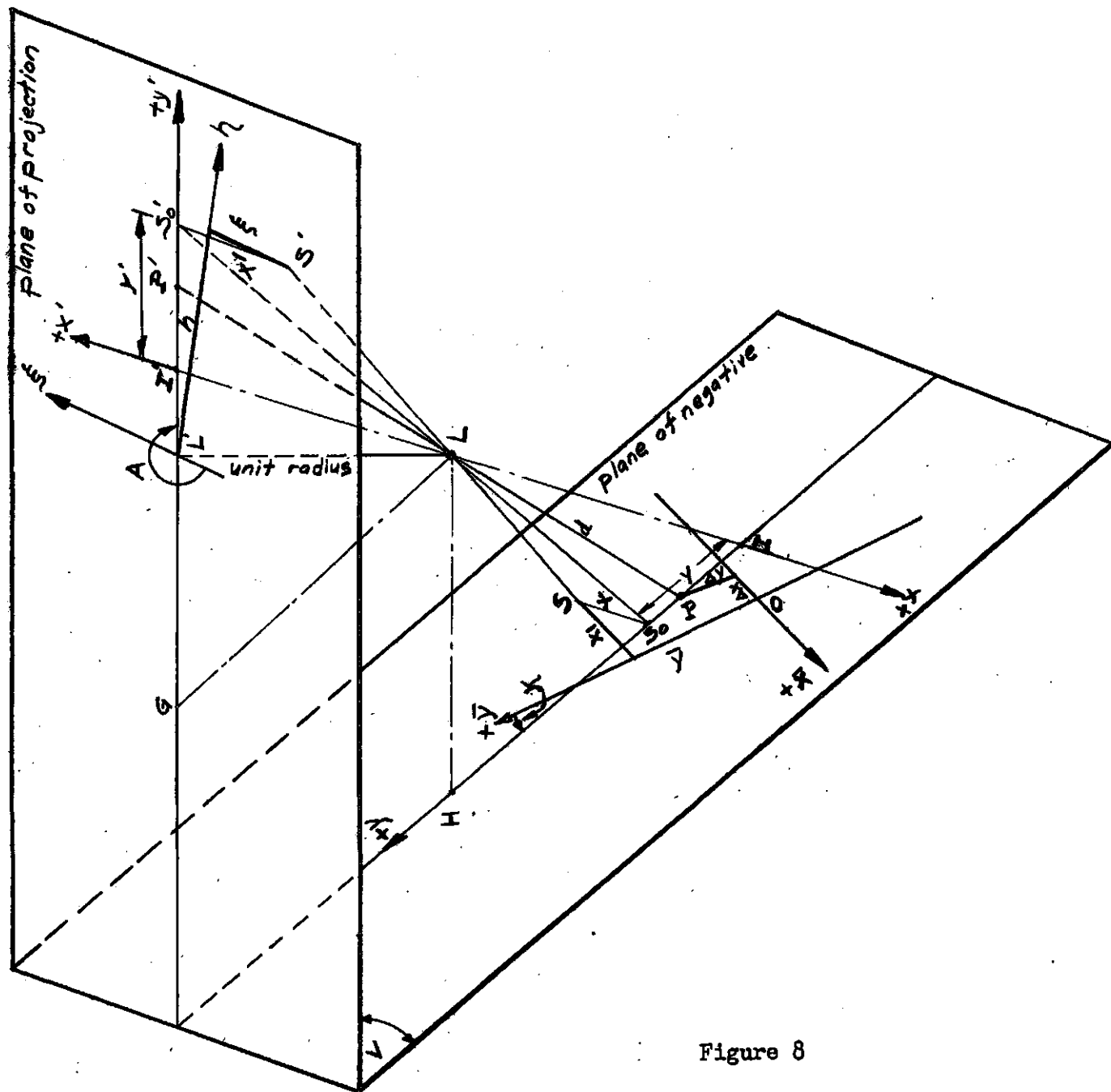


Figure 8

- \bar{x}, \bar{y} Primary plate coordinates. Plane rectangular coordinates of an image point, related to a system, whose origin (0) is the intersection of the \bar{x} and \bar{y} axes, established by fiducial marks.
- x, y Oriented plate coordinates. Plane, rectangular coordinates of an image point. The y -axis is the line of intersection of the image plane with a plane perpendicular to the plane of projection which contains the plate perpendicular. The origin is the isocenter I. The x, y -system is tilted against the \bar{x}, \bar{y} -system by the swing-angle K (K counted clockwise).
- x', y' Plane rectangular coordinates of a star in the plane of projection. The system corresponds in its orientation to the x, y system. The origin is the corresponding isocenter I' . x_0 and y_0 denote the coordinates of the origin of the \bar{x}, \bar{y} system (0) in the x, y system.

A = azimuth of the plate perpendicular

v = tilt of the plate perpendicular

d = principal distance.

From Figures 8 and 9, we obtain again:

$$HI = HL = \frac{d}{\sin v} = d' \quad (4c)$$

$$GI' = GL = \frac{h}{\sin v} = h'$$

$\triangle GLS_0' \sim \triangle HS_0L$, therefore

$$\frac{GS_0'}{GL} = \frac{HL}{HS_0} \quad (5c)$$

and

$$\frac{x'}{x} = \frac{LS_0'}{LS_0} = \frac{GS_0'}{HL} = \frac{GL}{HS_0} \quad (6a)$$

Since $HS_0 = LH - y$ and $GS_0' = GL + y'$ we obtain again, from formulas 4, 5 and 6,

$$y' = \frac{hy}{d-y \sin v} \quad \text{and} \quad x' = \frac{hx}{d-y \sin v} \quad (7c)$$

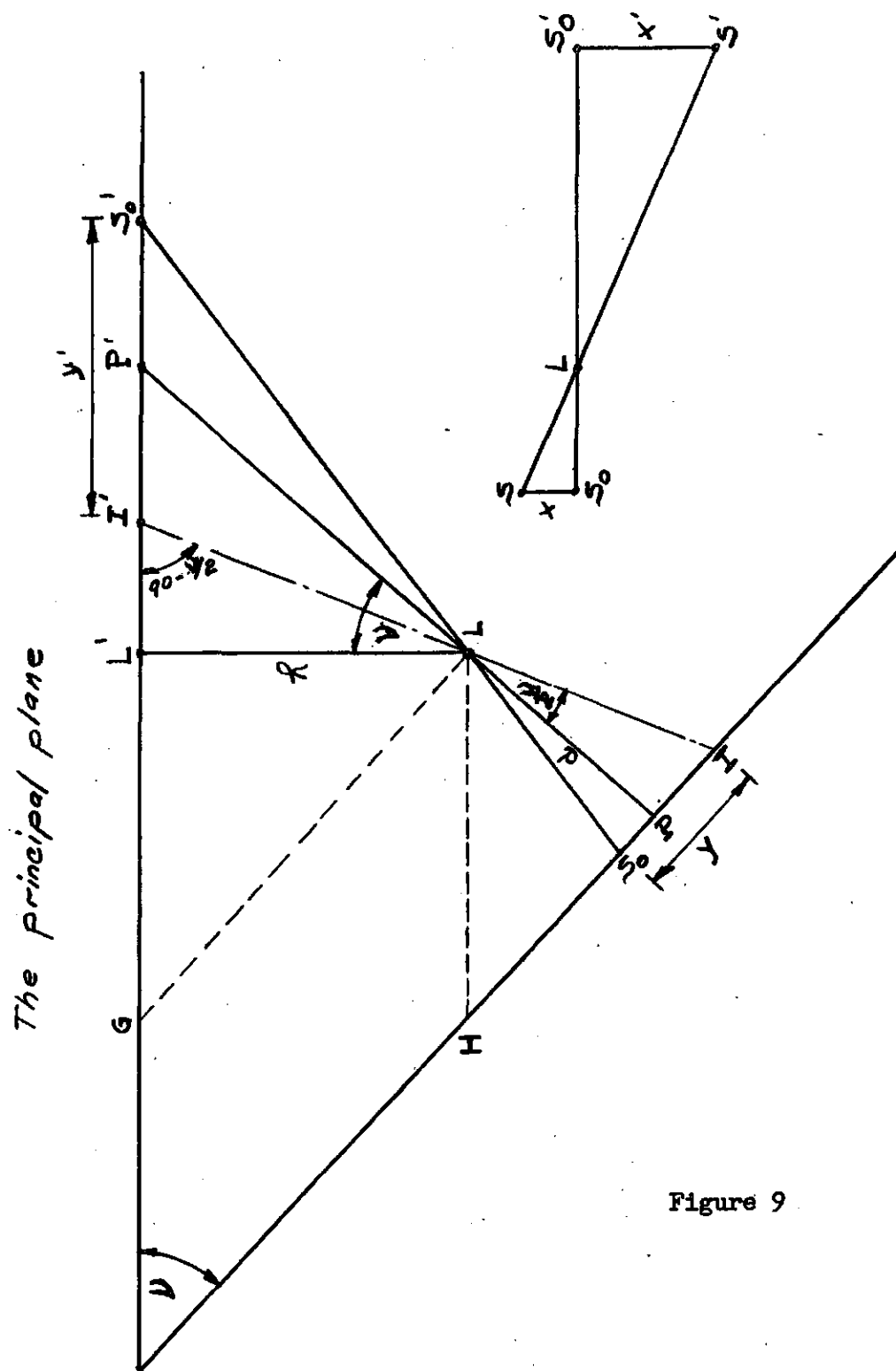


Figure 9

Furthermore:

$$\begin{aligned}\xi &= -x' \sin A - y' \cos A + \xi_i \\ \eta &= -y' \sin A + x' \cos A + \eta_i\end{aligned}\quad (11)$$

where ξ_i, η_i are the standard coordinates of I' in the ξ, η system, and

$$x = \bar{x} \cos \kappa + \bar{y} \sin \kappa + x_0 \quad (12)$$

$$y = \bar{y} \cos \kappa - \bar{x} \sin \kappa + y_0$$

where x_0, y_0 are the coordinates of O in the x, y system.

For $h = 1$ and by means of formula 7₆ we obtain from formula 11:

$$\begin{aligned}\xi &= \frac{-x \sin A - y \cos A}{d - y \sin v} + \eta_i \\ \eta &= \frac{-y \sin A + x \cos A}{d - y \sin v} + \xi_i\end{aligned}\quad (13)$$

and from Formula (12):

$$\begin{aligned}\xi &= \frac{a_1 \bar{x} + b_1 \bar{y} + c_1}{a_0 \bar{x} + b_0 \bar{y} + 1} \\ \eta &= \frac{a_2 \bar{x} + b_2 \bar{y} + c_2}{a_0 \bar{x} + b_0 \bar{y} + 1}\end{aligned}\quad (14)$$

where

$$\left. \begin{aligned}a_1 &= \frac{-h' \sin (A-\kappa) + \xi_i \sin \kappa}{d' - y_0} \\ b_1 &= \frac{-h' \cos (A-\kappa) - \xi_i \cos \kappa}{d' - y_0} \\ a_2 &= \frac{+h' \cos (A-\kappa) + \eta_i \sin \kappa}{d' - y_0} \\ b_2 &= \frac{-h' \sin (A-\kappa) - \eta_i \cos \kappa}{d' - y_0}\end{aligned} \right\} (15)$$

$$c_1 = \frac{-h'(x_0 \sin A + y_0 \cos A)}{d' - y_0} + \xi_1$$

$$c_2 = \frac{-h'(y_0 \sin A - x_0 \cos A)}{d' - y_0} + \eta_1$$

(15)
cont'd

$$a_0 = -\frac{\sin K}{d' - y_0}$$

$$a_0' = -\frac{\sin K}{d' - y_0}$$

$$b_0 = -\frac{\cos K}{d' - y_0}$$

$$b_0' = -\frac{\cos K}{d' - y_0}$$

and the elements of orientation may be computed by:

$$\tan K = \frac{-a_0}{-b_0}$$

$$\xi_1 = \frac{-a_0(a_2 + b_1) + b_0(a_1 - b_2)}{a_0^2 + b_0^2}$$

$$d' - y_c' = \frac{-\sin K}{a_0} = \frac{-\cos K}{b_0} = \frac{1}{\sqrt{a_0^2 + b_0^2}} \quad \eta_1 = \frac{-a_0(a_1 - b_2) - b_0(a_2 + b_1)}{a_0^2 + b_0^2}$$

$$\cot(A+K) = \frac{a_0(b_0 a_1 - a_0 b_1) - b_0(a_0 b_2 - b_0 a_2)}{-b_0(b_0 a_1 - a_0 b_1) - a_0(a_0 b_2 - b_0 a_2)}$$

$$h' = \frac{a_0(b_0 a_1 - a_0 b_1) - b_0(a_0 b_2 - b_0 a_2)}{+ \cos(A+K)} (d' - y_c')^3$$

$$= \frac{-b_0(b_0 a_1 - a_0 b_1) - a_0(a_0 b_2 - b_0 a_2)}{+ \sin(A+K)} (d' - y_c')^3$$

Or introducing ξ_1 and η_1

$$\cot(A+K) = \frac{-b_0 \eta_1 - b_1}{-a_0 \eta_1 - a_1} = \frac{a_0 \xi_1 + a_2}{-b_0 \xi_1 - b_2}$$

And

$$h' = \frac{(-a_0 \eta_1 - a_1)(d' - y_c')}{+ \sin(A+K)} = \frac{(-b_0 \eta_1 - b_1)(d' - y_c')}{+ \cos(A+K)}$$

$$= \frac{(a_0 \xi_1 + a_2)(d' - y_c')}{+ \cos(A+K)} = \frac{(-b_0 \xi_1 - b_2)(d' - y_c')}{+ \sin(A+K)}$$

(16)

$$x_o' = \left[-(c_1 + \eta_1) \sin A + (c_2 + \xi_1) \cos A \right] \frac{d' - y_c'}{h'}$$

$$y_c' = \left[-(c_2 + \xi_1) \sin A - (c_1 + \eta_1) \cos A \right] \frac{d' - y_c'}{h'}$$

(16)

cont'd

$$d' = (d' - y_c') + y_c'$$

$$d = d' \sin \nu$$

$$\sin \nu = \frac{1}{h'}$$

$$y_o' = d \tan \frac{\nu}{2}$$

$$\Delta x = -x_c' \cos K + (y_o' - y_c' \sin K$$

$$\Delta y = (y_o' - y_c') \cos K + x_c' \sin K$$

The formulas (14) again express the relation between the standard coordinates ξ, η and the plate coordinates \bar{x}, \bar{y} in any rectangular plate coordinate system. The unknowns of the solution are the so-called plate constants a_1, b_1, c_1, a_0, b_0 and a_2, b_2, c_2, a', b' . Formulas based on plate constants¹ were used in earlier work on ballistic cameras and related problems and are described in various BRL Reports.*

- * 1. R. S. Zug, "A Photometeroronic Method for Bomb Ballistics and for Measurements of the Flight Performance of Aircraft," BRL Report 279, p.35.
- 2. T. E. Sterne, "The Precise Determination of the Position of a Point in Space from Photographs taken at Two Ground Stations," BRL Report 273, p. 14.
- 3. R. S. Zug, "High Altitude Range Bombing by the Aberdeen Bombing Mission Using Ballistic Cameras", LSD-Report 1, pp. 40, 43.
- 4. L. G. deBey and R. S. Zug, "History of the Ballistic Camera and Range Bombing Instrumentation", BRL Report 577, p. 2.
- 5. R. d'E Atkinson, "A Shortened Method of Star-reductions for Calibrating Ballistic Cameras, BRL Report 593, p. 12.
- 6. Virginia Farquar, "A Short Method for Reduction of Ballistic Camera Observations", BRL Technical Note 50, p. 2.
- 7. Dirk Reuyl, "Full-Scale Free-Flight Ballistic Measurements of Aircraft Rockets by Optical Methods", BRL Report 676, p. 40.

A comparison between formulas (14), (10a) and (10b), which obviously express the same relations shows that six unknowns are present in formulas (10a) and (10b) and ten unknowns in formula 14. We have already seen that six unknowns are necessary and sufficient to solve the problem geometrically.

Consequently, the ten unknowns in formula (14) are not all independent parameters, but four additional condition equations must exist between at least some of the unknowns.* A first examination of the terms expressing the plate constants in formulas (15) shows that

$$1) \quad a_0 \equiv a_0'$$

$$2) \quad b_0 \equiv b_0'$$

The other two condition equations can hardly be eliminated from the given relations. However, it is possible to write the equations (10a) in the following arrangement:

$$\begin{aligned} & \bar{y}(-\cos \kappa \cos v \cos A + \sin \kappa \sin A) + \bar{x}(-\cos \kappa \sin A - \sin \kappa \cos v \cos A) + \\ \xi = & \frac{\Delta y(\cos \kappa \cos v \cos A - \sin \kappa \sin A) + \Delta x(\cos \kappa \sin A + \sin \kappa \cos v \cos A) - d \sin v \cos A}{-\bar{y} \cos \kappa \sin v - \bar{x} \sin \kappa \sin v + \Delta y \cos \kappa \sin v + \Delta x \sin \kappa \sin v + d \cos v} \end{aligned}$$

$$\begin{aligned} & \bar{y}(-\cos \kappa \cos v \sin A - \sin \kappa \cos A) + \bar{x}(\cos \kappa \cos A - \sin \kappa \cos v \sin A) + \\ \eta = & \frac{\Delta y(\cos \kappa \cos v \sin A + \sin \kappa \cos A) + \Delta x(-\cos \kappa \cos A + \sin \kappa \cos v \sin A) - d \sin v \sin A}{-\bar{y} \cos \kappa \sin v - \bar{x} \sin \kappa \sin v + \Delta y \cos \kappa \sin v + \Delta x \sin \kappa \sin v + d \cos v} \end{aligned}$$

or

$$\xi = \frac{a_1 \bar{x} + b_1 \bar{y} + c_1}{a_0 \bar{x} + b_0 \bar{y} + 1}$$

$$\eta = \frac{a_2 \bar{x} + b_2 \bar{y} + c_2}{a_0 \bar{x} + b_0 \bar{y} + 1}$$

where

$$a_1 = \frac{-\cos \kappa \sin A \sec v - \sin \kappa \cos A}{\lambda}$$

} (17a)

* See BRL Memorandum Report 476 by S. T. Zaroodny, "On the Use of Least Squares in the Determination of Plate Constants."

$$b_1 = \frac{+ \sin \kappa \sin A \sec v - \cos \kappa \cos A}{\lambda}$$

$$c_1 = \frac{-d \tan v \cos A + \Delta x (\cos \kappa \sin A \sec v + \sin \kappa \cos A) + \Delta y (-\sin \kappa \sin A \sec v + \cos \kappa \cos A)}{\lambda} \quad (17a)$$

cont'd

$$a_2 = \frac{+ \cos \kappa \cos A \sec v - \sin \kappa \sin A}{\lambda}$$

$$b_2 = \frac{- \sin \kappa \cos A \sec v - \cos \kappa \sin A}{\lambda}$$

$$c_2 = \frac{-d \tan v \sin A + \Delta x (-\cos \kappa \cos A \sec v + \sin \kappa \sin A) + \Delta y (\sin \kappa \sin A \sec v + \cos \kappa \sin A)}{\lambda}$$

$$a_0 = \frac{- \tan v \sin \kappa}{\lambda}$$

$$b_0 = \frac{- \tan v \cos \kappa}{\lambda}$$

where $\lambda = d + \tan v (\Delta y \cos \kappa + \Delta x \sin \kappa)$.

$$\text{Therefore } \tan \kappa = \frac{-a_0}{-b_0} \quad \text{and } \sin 2A = \frac{2(a_1 a_2 + b_1 b_2)}{a_0^2 + b_0^2} \quad (18)$$

and after suitable transformation, we obtain the following independent condition equations:

$$1) \quad a_0 = a_0'$$

$$2) \quad b_0 = b_0'$$

$$3) \quad a_0 b_0 + a_1 b_1 + a_2 b_2 = 0 \quad (19)$$

$$4) \quad a_0^2 + a_1^2 + a_2^2 - b_0^2 - b_1^2 - b_2^2 = 0$$

Using formula 10^b, the plate constants are given by the following expressions:

$$\begin{aligned}
 a_1 &= \frac{\sin \kappa \sec \omega - \cos \kappa \tan \omega \tan \alpha}{\lambda} \\
 b_1 &= \frac{\cos \kappa \sec \omega + \sin \kappa \tan \omega \tan \alpha}{\lambda} \\
 c_1 &= \frac{-\Delta y (\sin \kappa \sec \omega - \cos \kappa \tan \omega \tan \alpha) - \Delta x (\cos \kappa \sec \omega + \sin \kappa \tan \omega \tan \alpha) + d \tan \alpha}{\lambda} \\
 a_2 &= \frac{\cos \kappa \sec \alpha}{\lambda} & a_0 &= \frac{-\sin \kappa \sec \omega \tan \alpha - \cos \kappa \tan \omega}{\lambda} \\
 b_2 &= \frac{-\sin \kappa \sec \alpha}{\lambda} & b_0 &= \frac{-\cos \kappa \sec \omega \tan \alpha + \sin \kappa \tan \omega}{\lambda} \\
 c_2 &= \frac{(-\Delta y \cos \kappa + \Delta x \sin \kappa + d \tan \omega) \sec \alpha}{\lambda}
 \end{aligned} \tag{17b}$$

where

$$\lambda = d + \Delta y (\sin \kappa \sec \omega \tan \alpha + \cos \kappa \tan \omega) + \Delta x (\cos \kappa \sec \omega \tan \alpha - \sin \kappa \tan \omega)$$

The condition equations may be proved by substituting the expressions for the plate constants obtained either from formulas 15, 17a or 17b.

Formulas 14 with the ten plate constants and the four condition equations (formulas 19) represent now a system of six independent parameters. Now, all three results, obtained from the two methods are consistent and in agreement with the theory of the orientation problem.

All three formula systems (10a, 10b and 14/19) express in rigorous geometrical or analytical terms the relation between the standard coordinates and the corresponding plate coordinates. Each of the three systems allows the computation of the plate orientation from three stars. In general, there will be more stars used than are necessary for a unique solution. In such a case, the result must be determined by a least squares solution, which may be based on any of the three systems of formulas.

3. The rigorous least squares adjustment

The formulas 10a, 10b, and 14, the last in combination with the condition equations 19, represent the rigorous mathematical relation between the plate coordinates measured in any plane rectangular system on the plate and denoted by \bar{x} and \bar{y} , and the standard coordinates ξ and η computed as plane rectangular coordinates in a plane tangent at a certain point on the unit sphere whose center is at the point of projection.

All three systems are therefore suitable for establishing the observation equations. We will consider these possibilities:

a) Projection method, based on the formulas 10a

The observations of the plate coordinates with reference to a fiducial marks-system are denoted by l and l' , corresponding to the \bar{x} and \bar{y} axes. The observational (residual) errors of these observations are denoted by v and v' ; hence, $\bar{x} = l + v$ and $\bar{y} = l' + v'$. Δx and Δy are the coordinates of the principal point (P) with reference to the fiducial mark system (\bar{x}, \bar{y}) whose origin is in O.

d = principal distance

A = azimuth of the optical axis

v = tilt of the optical axis

K = swing angle of the fiducial marks system

From formula 10a we obtain

$$\xi = \frac{(\bar{y} - \Delta y)(\cos K \cos v \cos A - \sin K \sin A) + (\bar{x} - \Delta x)(\sin K \cos v \cos A + \cos K \sin A) + d \sin v \cos A}{(\bar{y} - \Delta y) \cos K \sin v + (\bar{x} - \Delta x) \sin K \sin v - d \cos v} = F \quad (20)$$

$$\eta = \frac{(\bar{y} - \Delta y)(\cos K \cos v \sin A + \sin K \cos A) + (\bar{x} - \Delta x)(\sin K \cos v \sin A - \cos K \cos A) + d \sin v \sin A}{(\bar{y} - \Delta y) \cos K \sin v + (\bar{x} - \Delta x) \sin K \sin v - d \cos v} = F'$$

From the Taylor expansion for the right side of these equations we have, neglecting terms of second and higher order:

$$\xi = \xi_0 + \frac{\partial F}{\partial A} \Delta A + \frac{\partial F}{\partial v} \Delta v + \frac{\partial F}{\partial K} \Delta K + \frac{\partial F}{\partial d} \Delta d + \frac{\partial F}{\partial \Delta x} \Delta \Delta x + \frac{\partial F}{\partial \Delta y} \Delta \Delta y + \frac{\partial F}{\partial \bar{x}} v + \frac{\partial F}{\partial \bar{y}} v'$$

$$\eta = \eta_0 + \frac{\partial F'}{\partial A} \Delta A + \frac{\partial F'}{\partial v} \Delta v + \frac{\partial F'}{\partial K} \Delta K + \frac{\partial F'}{\partial d} \Delta d + \frac{\partial F'}{\partial \Delta x} \Delta \Delta x + \frac{\partial F'}{\partial \Delta y} \Delta \Delta y + \frac{\partial F'}{\partial \bar{x}} v + \frac{\partial F'}{\partial \bar{y}} v'$$

We have $\xi_c - \xi_o = \Delta \xi$

$$\eta_c - \eta_o = \Delta \eta$$

where ξ_c, η_c are standard coordinates computed from reference data.

Hence

$$-\frac{\partial F}{\partial x} v - \frac{\partial F}{\partial y} v' = \frac{\partial F}{\partial A} \Delta A + \frac{\partial F}{\partial v} \Delta v + \frac{\partial F}{\partial K} \Delta K + \frac{\partial F}{\partial d} \Delta d + \frac{\partial F}{\partial \Delta x} \Delta \Delta x + \frac{\partial F}{\partial \Delta y} \Delta \Delta y - \Delta \xi \quad (21)$$

$$-\frac{\partial F'}{\partial x} v - \frac{\partial F'}{\partial y} v' = \frac{\partial F'}{\partial A} \Delta A + \frac{\partial F'}{\partial v} \Delta v + \frac{\partial F'}{\partial K} \Delta K + \frac{\partial F'}{\partial d} \Delta d + \frac{\partial F'}{\partial \Delta x} \Delta \Delta x + \frac{\partial F'}{\partial \Delta y} \Delta \Delta y - \Delta \eta$$

We substitute now*

$$A = A_o + \Delta A$$

$$v = v_o + \Delta v$$

$$K = K_o + \Delta K$$

$$d = d_o + \Delta d$$

$$\Delta x = \Delta x_o + \Delta \Delta x; \Delta y = \Delta y_o + \Delta \Delta y$$

$$\bar{x} = l + v$$

$$\bar{y} = l' + v'$$

and introduce the following terms:

$$\xi_o = \frac{A(l' - \Delta y_o) + B(l - \Delta x_o) + C \cdot d_o}{D(l' - \Delta y_o) + E(l - \Delta x_o) + F \cdot d_o} = \frac{s}{u}$$

*For the computation of approximation values see page 50.

$$\eta_0 = \frac{A'(l' - \Delta y_0) + B'(l - \Delta x_0) + C' \cdot d_0}{D(l' - \Delta y_0) + E(l - \Delta x_0) + F \cdot d_0} = \frac{t}{u}$$

where

$$A = \cos K_0 \cos v_0 \cos A_0 - \sin K_0 \sin A_0 \quad A' = \cos K_0 \cos v_0 \sin A_0 + \sin K_0 \cos A_0$$

$$B = \sin K_0 \cos v_0 \cos A_0 + \cos K_0 \sin A_0 \quad B' = \sin K_0 \cos v_0 \sin A_0 - \cos K_0 \cos A_0$$

$$C = \sin v_0 \cos A_0 \quad C' = \sin v_0 \sin A_0$$

$$D = \cos K_0 \sin v_0$$

$$E = \sin K_0 \sin v_0$$

$$F = -\cos v_0$$

From formulas 20 and 21 we obtain after suitable transformations*

the observation equations:

$$\begin{aligned} \rho = \frac{-B + \xi_0 E}{u} v + \frac{-A + \xi_0 D}{u} v' = -\eta_0 \Delta A \\ - \left[(1 + \xi_0^2) \cos A_0 + \xi_0 \sin A_0 \right] \Delta v \\ - \left[(l - \Delta x_0) \frac{-A + \xi_0 D}{u} - (l' - \Delta y_0) \frac{-B + \xi_0 E}{u} \right] \Delta K \\ + \frac{-B + \xi_0 E}{u} \cdot \Delta \Delta x \\ + \frac{-A + \xi_0 D}{u} \Delta \Delta y \\ + \frac{C - \xi_0 F}{u} \Delta d \end{aligned} \quad (23)$$

$$\text{with the weight } p = \frac{u^2}{(-B + \xi_0 E)^2 + (-A + \xi_0 D)^2}$$

* The entirely analytical computation of some of the coefficients asks for rather complex transformations. The result, however, may be obtained directly by using vector analysis. Such a solution is shown in BRL Report No. 785, H. Schmid, "Error Theory of Intersection Photogrammetry."

$$\begin{aligned}
\rho' = & \frac{-B' + \eta_o E}{u} v + \frac{-A' + \eta_o D}{u} v' = + \xi_o \Delta A \\
& - \left[(1 + \eta_o^2) \sin A_o + \eta_o \cos A_o \right] \Delta v \\
& - \left[(1 - \Delta x_o) \frac{-A' + \eta_o D}{u} - (1 - \Delta y_o) \frac{-B' + \eta_o E}{u} \right] \Delta K \\
& + \frac{-B' + \eta_o E}{u} \Delta \Delta x \\
& + \frac{-A' + \eta_o D}{u} \Delta \Delta y \\
& + \frac{C' - \eta_o F}{u} \Delta d \\
& - \Delta \eta
\end{aligned}
\tag{23 cont'd}$$

$$\text{with the weight } p' = \frac{u^2}{(-B' + \eta_o E)^2 + (-A' + \eta_o D)^2}$$

The number of unknowns may be reduced by eliminating ΔA . We divide for this purpose each of the observation equations by the factor which is combined with ΔA thus obtaining reduced observation equations.

We introduce furthermore:

$$a = \frac{+B - \xi_o E}{t}$$

$$a' = \frac{-B' + \eta_o E}{s}$$

$$b = \frac{+A - \xi_o D}{t}$$

$$b' = \frac{-A' + \eta_o D}{s}$$

$$c = \frac{-C + \xi_o F}{t}$$

$$c' = \frac{+C' - \eta_o F}{s}$$

$$L = - \frac{\Delta \xi}{\eta_o}$$

$$L' = + \frac{\Delta \eta}{\xi_o}$$

Formula 23 may be written as follows:

$$\rho = a \cdot v \cdot b \cdot v' = \Delta A + \left[\frac{1 + \xi_0^2}{\eta_0} \cos A_0 + \xi_0 \sin A_0 \right] \Delta v - \left[(l - \Delta x_0) b - (l' - \Delta y_0) a \right] \Delta K$$

$$+ a \cdot \Delta \Delta x + b \cdot \Delta \Delta y + c \cdot \Delta d - L \quad (23')$$

$$\rho' = a' \cdot v \cdot b' \cdot v' = \Delta A' - \left[\frac{1 + \eta_0^2}{\xi_0} \sin A_0 + \eta_0 \cos A_0 \right] \Delta v - \left[(l - \Delta x_0) b' - (l' - \Delta y_0) a' \right] \Delta K$$

$$+ a' \cdot \Delta \Delta x + b' \cdot \Delta \Delta y + c' \cdot \Delta d - L'$$

or:

$$\rho = \Delta A + \alpha \Delta v + \beta \Delta K + a \cdot \Delta \Delta x + b \cdot \Delta \Delta y + c \cdot \Delta d - L$$

$$\rho' = \Delta A' + \alpha' \Delta v + \beta' \Delta K + a' \cdot \Delta \Delta x + b' \cdot \Delta \Delta y + c' \cdot \Delta d - L'$$

and with the help of the first normal equations, we obtain the reduced observation equations:

$$\rho = (A) \Delta v + (B) \Delta K + (C) \Delta \Delta x + (D) \Delta \Delta y + (E) \Delta d - (L) \quad (24)$$

$$\rho' = (A') \Delta v + (B') \Delta K + (C') \Delta \Delta x + (D') \Delta \Delta y + (E') \Delta d - (L')$$

$$\text{where } (A) = \left(\alpha - \frac{[\alpha + \alpha']}{2n} \right); \quad (A') = \left(\alpha' - \frac{[\alpha + \alpha']}{2n} \right)$$

$$(B) = \left(\beta - \frac{[\beta + \beta']}{2n} \right); \quad (B') = \left(\beta' - \frac{[\beta + \beta']}{2n} \right)$$

$$(C) = \left(a - \frac{[a + a']}{2n} \right); \quad (C') = \left(a' - \frac{[a + a']}{2n} \right)$$

$$(D) = \left(b - \frac{[b + b']}{2n} \right); \quad (D') = \left(b' - \frac{[b + b']}{2n} \right)$$

$$(E) = \left(c - \frac{[c + c']}{2n} \right); \quad (E') = \left(c' - \frac{[c + c']}{2n} \right)$$

$$(L) = \left(L - \frac{[L + L']}{2n} \right); \quad (L') = \left(L' - \frac{[L + L']}{2n} \right)$$

and

$$\Delta A = -\frac{[\alpha + \alpha']}{2n} \Delta v - \frac{[\beta + \beta']}{2n} \Delta K - \frac{[\alpha + \alpha']}{2n} \Delta \Delta x - \frac{[\beta + \beta']}{2n} \Delta \Delta y - \frac{[\gamma + \gamma']}{2n} \Delta d + \frac{[\lambda + \lambda']}{2n} \quad (25)$$

$$v = \frac{b'p - bp'}{ab' - a'b} \quad (26)$$

$$v' = \frac{ap' - a'p}{ab' - a'b}$$

The reduced observation equations (24) have different weights due to the fact that in each of the equations there is more than one observed quantity. The weights are:

$$p = \frac{1}{a^2 + b^2} \quad (27)$$

$$p' = \frac{1}{a'^2 + b'^2}$$

The normal equations are:

$$\begin{aligned} [p(AA)] \Delta v + [p(AB)] \Delta K + [p(AC)] \Delta \Delta x + [p(AD)] \Delta \Delta y + [p(AE)] \Delta d - [p(AL)] &= 0 \\ [p(BB)] \Delta K + [p(BC)] \Delta \Delta x + [p(BD)] \Delta \Delta y + [p(BE)] \Delta d - [p(BL)] &= 0 \\ [p(CC)] \Delta \Delta x + [p(CD)] \Delta \Delta y + [p(CE)] \Delta d - [p(CL)] &= 0 \\ [p(DD)] \Delta \Delta y + [p(DE)] \Delta d - [p(DL)] &= 0 \\ [p(EE)] \Delta d - [p(EL)] &= 0 \\ [p(LL)] &= 0 \end{aligned} \quad (28)$$

where

$[p(AA)]$ stands for $[p(A)(A) + p'(A')(A')]$ and the other terms in parentheses are denoted correspondingly.

After Δv , ΔK , $\Delta \Delta x$, $\Delta \Delta y$ and Δd are determined*, ΔA is obtained from equation 25. The residuals p and p' are computed from the reduced observation equations 24 and the residuals of the original plate measurements are obtained from formulas 26.

The final result is obtained by applying the formulas (23). Checks for the computations are:

$$[p] + [p'] = 0, [ppp + p'p'p'] = [vv] + [v'v'] = [p \cdot 11 \cdot 5] \quad (29)$$

The main check is made by means of formulas 10a. If the final elements of orientation and the corrected observations \bar{x} and \bar{y} are introduced, the computed ξ and η values must be in complete agreement with the originally obtained standard coordinates ξ_c and η_c respectively. In case of residuals which are caused by the neglected second order terms, the solution must be repeated by using the obtained results as new approximations. It will be sufficient in such a case to use the former coefficients in the normal equation system and to limit the additional computations to the computing of new L and L' terms.

The mean error of a measured plate coordinate may be determined by:

$$m = \pm \sqrt{\frac{[vv] + [v'v']}{2n - 6}} \quad (30)$$

and the mean errors of the unknowns are:

$$m_v = \pm \frac{m}{\sqrt{AA \cdot 4}}; \quad m_{\Delta x} = \pm \frac{m}{\sqrt{CC \cdot 4}}; \quad m_d = \pm \frac{m}{\sqrt{EE \cdot 4}} \quad (31)$$

$$m_K = \pm \frac{m}{\sqrt{BB \cdot 4}}; \quad m_{\Delta y} = \pm \frac{m}{\sqrt{DD \cdot 4}}$$

* For the reductions of the normal equations the "modernized" Gaussian Algorithm is suggested. The method is closely related to the Doolittle method and to Reicheneder's method. However, it is believed that the modernized Gaussian Algorithm combines a minimum of recording work with a maximum of protection against computing errors. See: Der "modernisierte" Gauss'sche Algorithmus zur Auflösung von Normalgleichungen, by H. Wolf in Zeitschrift für Vermessungswesen, #11 November 1950.

m_A may be determined from formula (25) by combining the individual mean errors. However, the mean errors are not independent and therefore cannot be propagated according to the Gaussian law of errors. The values $AA.4$, etc., have here the conventional meaning of the Gaussian reduction method.*

b. Projection method, based on the formula 10b.

The observations of the plate coordinates with reference to a fiducial marks system are again denoted by l and l' corresponding to the \bar{x} and \bar{y} axes. The observational errors of these observations are v and v' . Hence, $\bar{x} = l + v$ and $\bar{y} = l' + v'$. Δx and Δy are the coordinates of the principal point (P) with reference to the fiducial marks system (\bar{x}, \bar{y}) whose origin is in O. d denotes the principal distance, $\alpha = \xi$ -tilt angle and $\omega = \eta$ -tilt angle respectively. K denotes the swing angle of the fiducial marks system. From (10b) we obtain:

$$\xi = \frac{(\bar{y}-\Delta y)(\sin K \cos \alpha - \cos K \sin \omega \sin \alpha) + (\bar{x}-\Delta x)(\cos K \cos \alpha + \sin K \sin \omega \sin \alpha) + d \cos \omega \sin \alpha}{(\bar{y}-\Delta y)(-\cos K \sin \omega \cos \alpha - \sin K \sin \alpha) + (\bar{x}-\Delta x)(\sin K \sin \omega \cos \alpha - \cos K \sin \alpha) + d \cos \omega \cos \alpha} = \xi_0 \quad (32)$$

$$\eta = \frac{(\bar{y}-\Delta y) \cos K \cos \omega - (\bar{x}-\Delta x) \sin K \cos \omega + d \sin \omega}{(\bar{y}-\Delta y)(-\cos K \sin \omega \cos \alpha - \sin K \sin \alpha) + (\bar{x}-\Delta x)(\sin K \sin \omega \cos \alpha - \cos K \sin \alpha) + d \cos \omega \cos \alpha} = \eta_0$$

From the Taylor expansion for the right side of the above equations, we have, neglecting terms of second and higher order:

$$\xi = \xi_0 + \frac{\partial \xi}{\partial \alpha} \Delta \alpha + \frac{\partial \xi}{\partial \omega} \Delta \omega + \frac{\partial \xi}{\partial K} \Delta K + \frac{\partial \xi}{\partial d} \Delta d + \frac{\partial \xi}{\partial \Delta x} \Delta \Delta x + \frac{\partial \xi}{\partial \Delta y} \Delta \Delta y + \frac{\partial \xi}{\partial x} v + \frac{\partial \xi}{\partial y} v'$$

$$\eta = \eta_0 + \frac{\partial \eta}{\partial \alpha} \Delta \alpha + \frac{\partial \eta}{\partial \omega} \Delta \omega + \frac{\partial \eta}{\partial K} \Delta K + \frac{\partial \eta}{\partial d} \Delta d + \frac{\partial \eta}{\partial \Delta x} \Delta \Delta x + \frac{\partial \eta}{\partial \Delta y} \Delta \Delta y + \frac{\partial \eta}{\partial x} v + \frac{\partial \eta}{\partial y} v'$$

and with $\xi_c - \xi_0 = \Delta \xi$

(33)

$$\eta_c - \eta_0 = \Delta \eta$$

where ξ_c and η_c are standard coordinates computed from reference data,

* Ref: A Text Book on the Method of Least Squares by Mansfield Merriman, N.Y., 1911; or: Handbuch der Vermessungskunde-Ausgleichsrechnung by Jordan Eggert, volume I, Stuttgart, 1948.

$$- \frac{\partial G}{\partial x} v - \frac{\partial G}{\partial y} v' = \frac{\partial G}{\partial \alpha} \Delta \alpha + \frac{\partial G}{\partial \omega} \Delta \omega + \frac{\partial G}{\partial \kappa} \Delta \kappa + \frac{\partial G}{\partial d} \Delta d + \frac{\partial G}{\partial \Delta x} \Delta \Delta x + \frac{\partial G}{\partial \Delta y} \Delta \Delta y - \Delta \xi$$

$$- \frac{\partial G'}{\partial x} v - \frac{\partial G'}{\partial y} v' = \frac{\partial G'}{\partial \alpha} \Delta \alpha + \frac{\partial G'}{\partial \omega} \Delta \omega + \frac{\partial G'}{\partial \kappa} \Delta \kappa + \frac{\partial G'}{\partial d} \Delta d + \frac{\partial G'}{\partial \Delta x} \Delta \Delta x + \frac{\partial G'}{\partial \Delta y} \Delta \Delta y - \Delta \eta$$

We now substitute

$$\begin{aligned} \alpha &= \alpha_0 + \Delta \alpha \\ \omega &= \omega_0 + \Delta \omega \\ \kappa &= \kappa_0 + \Delta \kappa \\ d &= d_0 + \Delta d \\ \Delta x &= \Delta x_0 + \Delta \Delta x \\ \Delta y &= \Delta y_0 + \Delta \Delta y \\ \bar{x} &= l + v \\ \bar{y} &= l' + v' \end{aligned} \tag{34}$$

and introduce the following terms:

$$\begin{aligned} \xi_0 &= \frac{A(l' - \Delta y_0) + B(l - \Delta x_0) + C \cdot d_0}{D(l' - \Delta y_0) + E(l - \Delta x_0) + F \cdot d_0} = \frac{s}{u} \\ \eta_0 &= \frac{A'(l' - \Delta y_0) + B'(l - \Delta x_0) + C' \cdot d_0}{D(l' - \Delta y_0) + E(l - \Delta x_0) + F \cdot d_0} = \frac{t}{u} \end{aligned} \tag{35}$$

where:

$$\begin{aligned} A &= \sin \kappa_0 \cos \alpha_0 - \cos \kappa_0 \sin \omega_0 \sin \alpha_0 & A' &= \cos \kappa_0 \cos \omega_0 \\ B &= \cos \kappa_0 \cos \alpha_0 + \sin \kappa_0 \sin \omega_0 \sin \alpha_0 & B' &= -\sin \kappa_0 \cos \omega_0 \\ C &= \cos \omega_0 \sin \alpha_0 & C' &= \sin \omega_0 \end{aligned}$$

$$D = -\cos \kappa_0 \sin \omega_0 \cos \alpha_0 - \sin \kappa_0 \sin \alpha_0$$

$$E = \sin \kappa_0 \sin \omega_0 \cos \alpha_0 - \cos \kappa_0 \sin \alpha_0$$

$$F = \cos \omega_0 \cos \alpha_0$$

From the formula (32) we obtain the observation equations after suitable transformations*

$$\rho = \frac{-B + \xi_0 E}{u} v + \frac{-A + \xi_0 D}{u} v' = +(1 + \xi_0^2) \Delta \alpha$$

$$+(\eta_0 \xi_0 \cos \alpha_0 - \eta_0 \sin \alpha_0) \Delta \omega$$

$$+ \left[(l - \Delta x_0) \frac{-A + \xi_0 D}{u} - (l' - \Delta y_0) \frac{-B + \xi_0 E}{u} \right] \Delta \chi$$

$$+ \frac{-B + \xi_0 E}{u} \Delta \Delta x$$

$$+ \frac{-A + \xi_0 D}{u} \Delta \Delta y$$

$$+ \frac{C - \xi_0 F}{u} \Delta d$$

$$- \Delta \xi \text{ with the weight } p = \frac{u^2}{(-B + \xi_0 E)^2 + (-A + \xi_0 D)^2}$$

$$\rho' = \frac{-B' + \eta_0 E}{u} v + \frac{-A' + \eta_0 D}{u} v' = +\eta_0 \xi_0 \Delta \alpha$$

$$+ \left[(1 + \eta_0^2) \cos \alpha_0 + \xi_0 \sin \alpha_0 \right] \Delta \omega$$

$$+ \left[(l - \Delta x_0) \frac{-A' + \eta_0 D}{u} - (l' - \Delta y_0) \frac{-B' + \eta_0 E}{u} \right] \Delta \kappa$$

* See remark on page 34

$$+ \frac{-B' + \eta_o E}{u} \Delta \Delta x$$

$$+ \frac{-A' + \eta_o D}{u} \Delta \Delta y$$

$$+ \frac{C' - \eta_o F}{u} \Delta d$$

$$- \Delta \eta \text{ with the weight } p' = \frac{u^2}{(-B' + \eta_o E)^2 + (-A' + \eta_o D)^2}$$

The number of unknowns may be reduced by eliminating $\Delta \alpha$. For this purpose we divide each of the observation equations by the factor which is combined with $\Delta \alpha$ thus obtaining reduced observation equations.

We introduce furthermore:

$$a = \frac{-B + \xi_o E}{u + \xi_o \cdot s}$$

$$a' = \frac{-B' + \eta_o E}{\eta_o \cdot s}$$

$$b = \frac{-A + \xi_o D}{u + \xi_o \cdot s}$$

$$b' = \frac{-A' + \eta_o D}{\eta_o \cdot s}$$

$$c = \frac{+C - \xi_o F}{u + \xi_o \cdot s}$$

$$c' = \frac{+C' - \eta_o F}{\eta_o \cdot s}$$

Formula 36 may now be written as follows:

$$p = av + bv' = \Delta \alpha + \left[\frac{\eta_o \xi_o \cos \alpha_o - \eta_o \sin \alpha_o}{(1 + \xi_o^2)} \right] \Delta \omega + \left[(l - \Delta x_o) b - (l' - \Delta y_o) a \right] \Delta K +$$

$$a \Delta \Delta x + b \Delta \Delta y + c \Delta d - L \quad (36')$$

$$\rho' = a'v + b'v' = \Delta a + \left[\frac{1 + \eta_o^2}{\eta_o \xi_o} \cos \alpha_o + \frac{\sin \alpha_o}{\eta_o} \right] \Delta \omega + \left[(l - \Delta x)b' - (l' - \Delta y_o)a' \right] \Delta K +$$

$$a' \Delta \Delta x + b' \Delta \Delta y + c' \Delta d = L'$$

$$\text{where } L = \frac{\Delta \xi}{1 + \xi_o^2}$$

$$L' = \frac{\Delta \eta}{\eta_o \xi_o}$$

$$\text{The weight of the observation equations is } p = \frac{1}{a^2 + b^2}$$

(37)

$$p' = \frac{1}{a'^2 + b'^2}$$

The solution now follows the procedure outlined in the preceding paragraph.

c) Analytical Method, based on formulas 14 and 19

Substituting $a_o = a_o'$ and $b_o = b_o'$, formulas 14 become

$$a_1 \bar{x} + b_1 \bar{y} + c_1 - a_o \xi \bar{x} - b_o \xi \bar{y} - \xi = 0$$

and

$$a_2 \bar{x} + b_2 \bar{y} + c_2 - a_o \eta \bar{x} - b_o \eta \bar{y} - \eta = 0$$

(38)

We again introduce $\bar{x} = l + v$ and $\bar{y} = l' + v'$ and approximation values of the plate constants as follows

$$a_1 = a_1^o + \Delta a_1,$$

$$a_2 = a_2^o + \Delta a_2$$

$$b_1 = b_1^o + \Delta b_1,$$

$$b_2 = b_2^o + \Delta b_2$$

(39)

$$c_1 = c_1^0 + \Delta c_1,$$

$$c_2 = c_2^0 + \Delta c_2$$

$$a_0 = a_0^0 + \Delta a_0,$$

$$b_0 = b_0^0 + \Delta b_0,$$

We obtain from formulas 38, by the Taylor series, neglecting terms of second order and higher:

$$\begin{aligned} \rho = l \Delta a_1 + l' \Delta b_1 + \Delta c_1 - l \xi \Delta a_0 - l' \xi \Delta b_0 - \xi + l a_1^0 + l' b_1^0 + c_1^0 - l \xi a_0^0 - l' \xi b_0^0 = \\ v(\xi a_0^0 - a_1^0) + v'(\xi b_0^0 - b_1^0) \end{aligned} \quad (40)$$

$$\begin{aligned} \rho' = l \Delta a_2 + l' \Delta b_2 + \Delta c_2 - l \eta \Delta a_0 - l' \eta \Delta b_0 - \eta + l a_2^0 + l' b_2^0 + c_2^0 - l \eta a_0^0 - l' \eta b_0^0 = \\ v(\eta a_0^0 - a_2^0) + v'(\eta b_0^0 - b_2^0) \end{aligned}$$

or

$$\rho = l \Delta a_1 + l' \Delta b_1 + \Delta c_1 - l \xi_c \Delta a_0 - l' \xi_c \Delta b_0 - L, \text{ with weight } p \quad (41)$$

$$\rho' = l \Delta a_2 + l' \Delta b_2 + \Delta c_2 - l \eta_c \Delta a_0 - l' \eta_c \Delta b_0 - L', \text{ with weight } p'$$

$$p = \frac{1}{(\xi a_0^0 - a_1^0)^2 + (\xi b_0^0 - b_1^0)^2}; \quad p' = \frac{1}{(\eta a_0^0 - a_2^0)^2 + (\eta b_0^0 - b_2^0)^2}$$

The introduction of weighting factors becomes necessary because in each of the above derived observation equations more than one observation and correspondingly more than one residual appear.

$$-L = l a_1^0 + l' b_1^0 + c_1^0 - l \xi_c a_0^0 - l' \xi_c b_0^0 - \xi_c$$

$$-L' = l a_2^0 + l' b_2^0 + c_2^0 - l \eta_c a_0^0 - l' \eta_c b_0^0 - \eta_c$$

where the computed standard coordinates are denoted by ξ_c and η_c .

The condition equations (19) must be satisfied by the unknowns. The first two, namely $a \approx a'$ and $b \approx b'$ have already been introduced into the equations (38). The third and fourth conditions are:

$$3. a_0 b_0 + a_1 b_1 + a_2 b_2 = 0$$

$$4. a_0^2 + a_1^2 + a_2^2 - b_0^2 - b_1^2 - b_2^2 = 0$$

By means of the substitutions made in formula 38 and the Taylor series, neglecting terms of second and higher order, we obtain:

$$b_1^0 \Delta a_1 + a_1^0 \Delta b_1 + b_2^0 \Delta a_2 + a_2^0 \Delta b_2 + b_0^0 \Delta a_0 + a_0^0 \Delta b_0 + \lambda_1 = 0$$

$$\text{where } \lambda_1 = a_0 b_0 + a_1 b_1 + a_2 b_2^*$$

and

$$a_0^0 \Delta a_0 + a_1^0 \Delta a_1 + a_2^0 \Delta a_2 - b_0^0 \Delta b_0 - b_1^0 \Delta b_1 - b_2^0 \Delta b_2 + \lambda_2 = 0$$

$$\text{where } \lambda_2 = \frac{a_0^2 + a_1^2 + a_2^2 - b_0^2 - b_1^2 - b_2^2}{2}^*$$

After suitable transformation, we have

$$l \Delta a_1 + l' \Delta b_1 = + \Delta a_2 (l a' + l' b') + \Delta b_2 (l' a' - l b') + \Delta a_0 (l c' + l' d') + \Delta b_0 (l' c' - l d') + l e' + l' f' \quad (42)$$

* If the approximation values a_0^0 , a_1^0 , a_2^0 , b_0^0 , b_1^0 and b_2^0 are chosen in such a way that they satisfy the condition equations 3 and 4, both λ_1 and λ_2 become zero. At the end of this paragraph a procedure for obtaining such approximation values is shown. In routine reductions, where usually the preceding result will be introduced as the approximation, the condition equations will in general be sufficiently satisfied.

where

$$a' = \frac{-a_1^0 a_2^0 - b_1^0 b_2^0}{a_1^{0^2} + b_1^{0^2}}$$

$$b' = \frac{+b_1^0 a_2^0 - a_1^0 b_2^0}{a_1^{0^2} + b_1^{0^2}}$$

$$c' = \frac{-b_0^0 b_1^0 - a_0^0 a_1^0}{a_1^{0^2} + b_1^{0^2}}$$

$$d' = \frac{-a_1^0 b_0^0 + a_0^0 b_1^0}{a_1^{0^2} + b_1^{0^2}}$$

$$e' = \frac{-b_1^0 \lambda_1 - a_1^0 \lambda_2}{a_1^{0^2} + b_1^{0^2}}$$

$$f' = \frac{+b_1^0 \lambda_2 - a_1^0 \lambda_1}{a_1^{0^2} + b_1^{0^2}}$$

(43)

For $\lambda_1 = 0$ and $\lambda_2 = 0$, $e' = 0$ and $f' = 0$ and we obtain

$$\Delta a_1 = a' \Delta a_2 - b' \Delta b_2 + c' \Delta a_0 - d' \Delta b_0$$

(44)

$$\Delta b_1 = b' \Delta a_2 + a' \Delta b_2 + d' \Delta a_0 + c' \Delta b_0$$

Substituting 42 with $e' = 0$ and $f' = 0$ in the observation equations 40, we obtain the observation equations:

$$\Delta c_1 + (l a' + l' b') \Delta a_2 + (l' a' - l b') \Delta b_2 + (l c' + l' d' - l \xi_c) \Delta a_0 + (l' c' - l d' - l' \xi_c) \Delta b_0 - L = p$$

(45)

$$\Delta c_2 + l \Delta a_2 + l' \Delta b_2 - l \eta_c \Delta a_0 - l' \eta_c \Delta b_0 - L' = p'$$

or

$$\Delta c_1 + \alpha \Delta a_2 + \beta \Delta b_2 + \gamma \Delta a_0 + \delta \Delta b_0 - L = \rho \quad (46)$$

$$\Delta c_2 + l \Delta a_2 + l' \Delta b_2 - l \eta_c \Delta a_0 - l' \eta_c \Delta b_0 - L' = \rho'$$

where

$$\alpha = (l a' + l' b')$$

$$\beta = (l' a' - l b')$$

$$\gamma = [l(c' - \xi_c) + l' d'] ; \delta = [l'(c' - \xi_c) - l d']$$

Before we form normal equations we eliminate the unknowns Δc_1 and Δc_2 . Thus we obtain the reduced observation equations:

$$\left(\alpha - \frac{[a]}{n}\right) \Delta a_2 + \left(\beta - \frac{[\beta]}{n}\right) \Delta b_2 + \left(\gamma - \frac{[\gamma]}{n}\right) \Delta a_0 + \left(\delta - \frac{[\delta]}{n}\right) \Delta b_0 - \left(L - \frac{[L]}{n}\right) = \rho \quad (47)$$

$$\left(l - \frac{[l]}{n}\right) \Delta a_2 + \left(l' - \frac{[l']}{n}\right) \Delta b_2 - \left(l \eta_c - \frac{[l \eta_c]}{n}\right) \Delta a_0 - \left(l' \eta_c - \frac{[l' \eta_c]}{n}\right) \Delta b_0 - \left(L' - \frac{[L']}{n}\right) = \rho'$$

or

$$A \Delta a_2 + B \Delta b_2 + C \Delta a_0 + D \Delta b_0 - (L) = \rho \text{ with the weight } p \quad (48)$$

$$A' \Delta a_2 + B' \Delta b_2 + C' \Delta a_0 + D' \Delta b_0 - (L') = \rho' \text{ with the weight } p'$$

The corresponding normal equations are:

$$\begin{aligned} [pAA] \Delta a_2 + [pAB] \Delta b_2 + [pAC] \Delta a_0 + [pAD] \Delta b_0 - [pA(L)] &= 0 \\ [pBB] \Delta b_2 + [pBG] \Delta a_0 + [pBD] \Delta b_0 - [pB(L)] &= 0 \\ [pCG] \Delta a_0 + [pCD] \Delta b_0 - [pC(L)] &= 0 \\ [pDD] \Delta b_0 - [pD(L)] &= 0 \\ [p(L)(L)] &= 0 \end{aligned}$$

For these equations $[pAA]$ stands for $[pAA + p'A'A']$ and the other terms in parentheses are denoted correspondingly.

After the unknowns $\Delta a_2, \Delta b_2, \Delta a_0$ and Δb_0 are determined* we compute Δc_1 and Δc_2 by solving the equations:

$$\begin{aligned}\Delta c_1 &= -\frac{[\alpha]}{n} \Delta a_2 - \frac{[\beta]}{n} \Delta b_2 - \frac{[\gamma]}{n} \Delta a_0 - \frac{[\delta]}{n} \Delta b_0 + \frac{[\epsilon]}{n} \\ \Delta c_2 &= -\frac{[L]}{n} \Delta a_2 - \frac{[L']}{n} \Delta b_2 + \frac{[L\eta]}{n} \Delta a_0 + \frac{[L'\eta]}{n} \Delta b_0 + \frac{[L\eta]}{n}\end{aligned}\quad (50)$$

and from the equations (44) we compute Δa_1 and Δb_1 .

The final plate constants are obtained from the equations (39).

As a first check these values are introduced in the condition equations (formula 19), which must be satisfied. In case the condition equations are not sufficiently satisfied the solution must be repeated by using the results as new approximations. (Compare with p. 38) In such a case it is advisable to compute the a_1 and b_1 values with formulas 57 in order to obtain approximation values which satisfy the condition equations. The residuals ρ and ρ' are computed with formulas 48 and the final residuals of the plate measurements v and v' are obtained from (40) as follows:

$$\begin{aligned}v &= \frac{\rho(\eta b_0^0 - b_2^0) - \rho'(\xi b_0^0 - b_1^0)}{(\xi a_0^0 - a_1^0)(\eta b_0^0 - b_2^0) - (\eta a_0^0 - a_2^0)(\xi b_0^0 - b_1^0)} \\ v' &= \frac{\rho'(\xi a_0^0 - a_1^0) - \rho(\eta a_0^0 - a_2^0)}{(\xi a_0^0 - a_1^0)(\eta b_0^0 - b_2^0) - (\eta a_0^0 - a_2^0)(\xi b_0^0 - b_1^0)}\end{aligned}\quad (51)$$

A check is obtained by

$$[p] = [p'] = 0$$

and

$$[vv] + [v'v'] = [ppp] + [p'p'p'] = p(LL) \cdot 3 \quad (52)$$

The final check is made by means of formula 14. After the \bar{x} and \bar{y} values are obtained with $\bar{x} = l + v$ and $\bar{y} = l' + v'$ the corresponding standard coordinates ξ and η must be in complete agreement with the originally computed ξ_c and η_c . In addition the unknowns must satisfy the condition equations 19.

* Compare footnote on page 38.

The mean error of an observed plate coordinate is:

$$m = \pm \sqrt{\frac{[vv] + [v'v']}{2n - 6}} \quad (53)$$

and the mean errors of the unknowns may be computed from:

$$\begin{aligned} m_{a_2} &= \pm \frac{m}{\sqrt{AA \cdot 3}} \\ m_{b_2} &= \pm \frac{m}{\sqrt{BB \cdot 3}} \\ m_{a_0} &= \pm \frac{m}{\sqrt{CC \cdot 3}} \\ m_{b_0} &= \pm \frac{m}{\sqrt{DD \cdot 3}} \end{aligned} \quad (54)$$

The values $AA \cdot 3$, $BB \cdot 3$, etc., have the conventional meaning of the Gaussian reduction method. See footnote on page 39.

The mean errors of the other four plate constants denoted by m_{a_1} , m_{b_1} , m_{c_1} , and m_{c_2} may be obtained from the formulas 44 and 50. The correct computation is rather complicated. Since the individual mean errors of the unknowns are not independent, the computation must follow such lines as given in the computation of the mean error of a function of the unknowns.* For this purpose, the weight of these functions must be determined. If the corresponding weights are denoted by P_{a_1} , P_{b_1} , P_{c_1} , and P_{c_2} , the mean errors are:

$$\begin{aligned} m_{a_1} &= \pm m \sqrt{\frac{1}{P_{a_1}}} \\ m_{b_1} &= \pm m \sqrt{\frac{1}{P_{b_1}}} \\ m_{c_1} &= \pm m \sqrt{\frac{1}{P_{c_1}}} \\ m_{c_2} &= \pm m \sqrt{\frac{1}{P_{c_2}}} \end{aligned} \quad (55)$$

* Handbuch der Vermessungskunde Ausgleichsrechnung by Jordan-Eggert, Volume I - Stuttgart 1948 - p. 99

All three least squares solutions described assume that the reference data - i.e., the computed standard coordinates, are free of errors. The standard coordinates ξ and η are, in our case, functions of the right ascension and declination values which, in turn, are results obtained from observations and therefore affected by uncertainties which may be expressed in the form of mean errors. If the propagated mean errors of the standard coordinates are not negligible in relation to the expected mean errors of the plate measurements, it becomes necessary to make allowance for corrections to the computed standard coordinates. In such a case, we introduce $\xi = \xi_c + v_\xi$ and $\eta = \eta_c + v_\eta$, respectively. Such a step calls for additional weighting factors. Unit weight of an individual plate measurement may be assumed for both the \bar{x} and \bar{y} measurements, or $p_x = p_y = \frac{k}{m^2} = 1$, where m is the expected mean error of the \bar{x} and \bar{y} measurements. If the propagated mean errors of the standard coordinates are denoted by m_ξ and m_η the corresponding weights will be:

$$p_\xi = \frac{k}{(m_\xi \cdot d)^2} = \frac{m^2}{m_\xi^2 \cdot d^2} \quad (56)$$

$$p_\eta = \frac{k}{(m_\eta \cdot d)^2} = \frac{m^2}{m_\eta^2 \cdot d^2}$$

The fact that the orientation of the ξ, η system does not necessarily correspond to the orientation of the x, y system must be considered in computing the corrections v_ξ and v_η .

The Computation of Approximation Values for the Unknowns.

Approximation values for the unknowns are needed in all three methods of adjustment. In the projection method, approximation values of the orientation elements, and in the analytical method approximation values of the plate constants are necessary. The latter, in addition, should satisfy the existing condition equations (19). The relation between the orientation elements and the plate constants and their inverse transformation are given by formulas 15, 16, 17a and 17b.

In routine reductions, approximation values will generally be available from the results of the preceding reduction. In the case that none are available, the following procedure may be followed: With five stars well distributed over the plate, the plate constants are determined from formula (38). From the first equation, we obtain a_1, b_1, c_1, a_0' and b_0' and from the second equation, we compute a_2, b_2, c_2, a_0'' and b_0'' . The a_0' and a_0'' values and the b_0' and b_0'' values, respectively, are combined, by forming the arithmetic averages:

$$a_o = \frac{a_o' + a_o''}{2}$$

and

$$b_o = \frac{b_o' + b_o''}{2}$$

It is obvious that these approximations will not satisfy the condition equations (19). As a next step, therefore, we compute a set of orientation elements using formulas (16). These values may be taken directly as approximation values for the adjustment of the projection method. If the measuring camera is well adjusted, it normally will be sufficient to take zero as the approximation values for the coordinates of the principal point. Approximation values for the plate constants may now be computed by applying formulas 17a or 17b. The final approximation plate constants should be checked by the condition equations 19, before they are introduced in the least squares adjustment of the analytical method. For the analytical method a set of approximation values, consistent with the condition equations 19 may be directly computed with the help of the equations

$$a_1 = \frac{-a_2 b_2 - a_o b_o}{b_1}$$

with

$$b_1 = \sqrt{\frac{a_o^2 + a_2^2 - b_o^2 - b_2^2}{2}} \pm \sqrt{\left(\frac{a_o^2 + a_2^2 - b_o^2 - b_2^2}{2}\right)^2 + a_o^2 b_o^2 + a_2^2 b_2^2 + 2a_o b_o a_2 b_2}$$

The method of least squares adjustment which should be applied depends on the particular situation. In the case where all six elements of orientation are unknown, the analytic method seems to be advantageous, because the least squares adjustment can be arranged in such a way that only four normal equations need be reduced. However, there may be the case in which some orientation elements are known, e.g., from a preceding calibration procedure; thus calling only for the determination of the remaining unknown elements. The formulas given in the least squares methods for the projection method, are then preferred, because some of the unknowns will be equal to zero. If phototheodolites are used, where the approximation values are obtained from dial readings, the projection method offers a decided advantage.

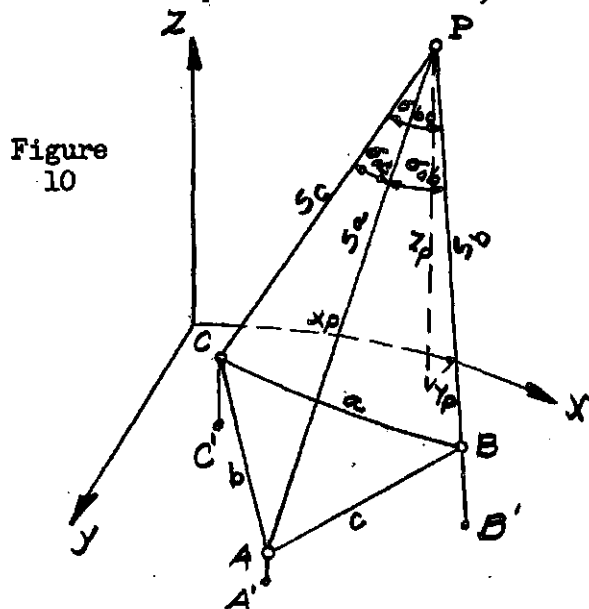
IV. THE MATHEMATICAL ANALYSIS OF THE SPATIAL TRIANGULATION PROBLEM

The theoretical background of the triangulation problem is explained in Chapter II. There it is shown that we have to deal with the two following triangulation methods:

- 1) The rigorous solution in triangulating a point from independent fixed photogrammetric cameras calls for resection in space.
- 2) After "calibration of the photogrammetric base line" the triangulation may be obtained by spatial intersection.

The latter method has been described in many reports. A method of least squares adjustment, suitable for combining two or more photogrammetric measuring results for the purpose of triangulation by intersection, is outlined in BRL Report 752*. However, it must be understood that although the described method provides a rigorous least squares solution for triangulation problems based on theodolite data, in our case it is only an approximation method due to the fact that the angular corrections and not the corrections to the original plate coordinate measurements are minimized.

We will, in this report, deal only with the resection problem. From the spatial coordinates of the measuring stations, denoted by x_i , y_i , z_i , which may be available in any coordinate system, the slant distance between any two of such stations can be computed. Assuming the three camera stations A, B, and C, we have the slant distances a, b, and c between these stations. (Fig. 10) From the least squares adjustment of the plate orientation, we have for each station a set of unknowns, either plate constants or orientation elements, depending on which method was applied. Introducing the measured plate coordinates for an additional recorded target point, after suitable corrections for non-perpendicularity of the comparator axes and corrections for lens distortion are applied, the standard coordinates ξ and η for such a target point are determined with the formulas 10a, 10b, or 11 and the corresponding direction angles may be found by solving the equations 1 or 2.



* BRL Report 752 by H. Schmid. Title: Spatial Triangulation by Least Squares Adjustment of Conditioned Observations.

From equation 1 for the resection method:

$$\cot \delta = \sqrt{\eta^2 + \xi^2} \quad (58)$$

$$\tan RA = \frac{-\xi}{\eta}$$

and from equation 2 for the intersection method:

$$\tan z = \sqrt{\eta^2 + \xi^2} \quad (59)$$

$$\tan A = \frac{-\xi}{-\eta}$$

The spatial angles σ (Fig. 4 and 10) at the apex of the pyramid ABCP may be computed by the following formulas:

$$\begin{aligned} \cos \sigma_{ab} &= \sin \delta_{a_r} \sin \delta_{b_r} + \cos \delta_{a_r} \cos \delta_{b_r} \cos \Delta RA_{ab_r} \\ \cos \sigma_{ac} &= \sin \delta_{a_r} \sin \delta_{c_r} + \cos \delta_{a_r} \cos \delta_{c_r} \cos \Delta RA_{ac_r} \\ \cos \sigma_{bc} &= \sin \delta_{b_r} \sin \delta_{c_r} + \cos \delta_{b_r} \cos \delta_{c_r} \cos \Delta RA_{bc_r} \end{aligned} \quad (60)$$

The subscript "r" indicates that the angles are corrected for refraction. (See the remark about refraction on page 15. However, either astronomical or terrestrial refraction must be considered depending on the spatial position of the target point.) A difference in time (converted to sidereal time) of the orientation exposures at the different stations goes into the solution as a corresponding correction to the right ascension values. Our problem is now to determine the coordinates of a point $P(x_p, y_p, z_p)$

(Fig. 10). First it becomes necessary to compute the length of the sides of the pyramid denoted by s_a, s_b, s_c , and finally the coordinates of the point by the intersection of three spheres. The second part is a familiar problem for a triangulation procedure using length measuring methods, such as Doppler, etc. A possible solution is outlined in BRL Report 748.* Hence we are left with the problem to determine the lengths of the sides of a pyramid, given the length of the sides of the base triangle and the three angles at the apex of the pyramid. This is the basic problem of the resection in space. For the three triangular faces of the pyramid, we have the equations:

* BRL Report 748, H. Schmid. Title: A Rectangular Ellipsoidal Coordinate System for Trajectory Measurements. pp.14-17, pp.23-25.

$$a^2 = s_b^2 + s_c^2 - 2 s_b s_c \cos \sigma_{bc}$$

$$b^2 = s_a^2 + s_c^2 - 2 s_a s_c \cos \sigma_{ac} \quad (61)$$

$$c^2 = s_a^2 + s_b^2 - 2 s_a s_b \cos \sigma_{ab}$$

The further treatment of these equations results in an equation of the fourth degree which is not especially suitable for numerical computations. In view of the fact that an economical computation of the final coordinates by the intersection of three spheres calls for the introduction of approximation values of the spatial position of P, it seems advisable to follow the following procedure:

1) Suitable approximation values for the target position denoted by x_p^0 , y_p^0 , and z_p^0 must be obtained. Such values may be computed by using the intersection method for any two stations. The necessary azimuth and elevation angles may be computed from the declination and right ascension values with the sidereal time and the geographic coordinates of the station by well-known formulas of spherical astronomy.*

$$\sin \epsilon = \sin \phi \sin \delta + \cos \phi \cos \delta \cos t \quad (62)$$

$$\sin A = \frac{\cos \delta \sin t}{\cos \epsilon}$$

t = hour angle = sidereal time (G) - Right Ascension (RA)

With the approximation values for the coordinates of point P and the coordinates of the stations the corresponding approximation values for the sides of the pyramids (s_i^0) may be computed.

Applying the Taylor series to the formulas (61) and neglecting all terms of second and higher order, we obtain:

$$\begin{aligned} s_a &= s_a^0 + \Delta s_a \\ s_b &= s_b^0 + \Delta s_b \\ s_c &= s_c^0 + \Delta s_c \end{aligned} \quad (63)$$

* See references on page 15 and formulas at the end of this chapter.

$$(s_a^0 - s_b^0 \cos \sigma_{ab}) \Delta s_a + (s_b^0 - s_a^0 \cos \sigma_{ab}) \Delta s_b = -\frac{s_a^0{}^2 + s_b^0{}^2 - c^2}{2} + s_a^0 s_b^0 \cos \sigma_{ab}$$

$$(s_a^0 - s_c^0 \cos \sigma_{ac}) \Delta s_a + (s_c^0 - s_a^0 \cos \sigma_{ac}) \Delta s_c = -\frac{s_a^0{}^2 + s_c^0{}^2 - b^2}{2} + s_a^0 s_c^0 \cos \sigma_{ac}$$

$$(s_b^0 - s_c^0 \cos \sigma_{bc}) \Delta s_b + (s_c^0 - s_b^0 \cos \sigma_{bc}) \Delta s_c = -\frac{s_b^0{}^2 + s_c^0{}^2 - a^2}{2} + s_b^0 s_c^0 \cos \sigma_{bc}$$

or

$$A_1 \Delta s_a + B_1 \Delta s_b = C_1$$

$$A_2 \Delta s_a + B_2 \Delta s_c = C_2$$

$$A_3 \Delta s_b + B_3 \Delta s_c = C_3$$

and therefore

$$\Delta s_a = \frac{|D_{\Delta s_a}|}{|D|}$$

$$\Delta s_b = \frac{|D_{\Delta s_b}|}{|D|}$$

$$\Delta s_c = \frac{|D_{\Delta s_c}|}{|D|}$$

where

$$|D_{\Delta s_a}| = \begin{vmatrix} C_1 & B_1 & 0 \\ C_2 & 0 & B_2 \\ C_3 & A_3 & B_3 \end{vmatrix}$$

$$|D_{\Delta s_b}| = \begin{vmatrix} A_1 & C_1 & 0 \\ A_2 & C_2 & B_2 \\ 0 & C_3 & B_3 \end{vmatrix} \quad (64)$$

$$|D_{\Delta s_c}| = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & 0 & C_2 \\ 0 & A_3 & C_3 \end{vmatrix}$$

$$|D| = \begin{vmatrix} A_1 & B_1 & 0 \\ A_2 & 0 & B_2 \\ 0 & A_3 & B_3 \end{vmatrix}$$

The final length of the sides of the pyramid are determined by the formulas (63). The determination of the final coordinates now follows the procedure outlined in BRL Report 748.*

During the orientation and the triangulation computations, there is sometimes a need for direct or inverse transformation of the elements of orientation between the Right-Ascension-Declination system and the Azimuth-Elevation System.

The following formulas may be used for this transformation:

d = principal distance

$\left. \begin{matrix} \Delta x \\ \Delta y \end{matrix} \right\}$ = Coordinates of the principal point

K = Swing angle of the plate coordinate system

v = zenith-distance of the plate perpendicular

A = Astronomical azimuth of the plate perpendicular
counted clockwise from the south

RA = Right Ascension of the plate perpendicular
counted counter-clockwise

δ = Declination of the plate perpendicular

ξ, η = Standard coordinates of a star in the azimuth-
elevation system

ξ, η^* = Standard coordinates of a star in the RA-declination
system

* See reference on page 53 .

THE ELEMENTS OF ORIENTATION

In the Right Ascension-Declination System (denoted by *)

$$d^* = d$$

$$\Delta x^* = \Delta x$$

$$\Delta y^* = \Delta y$$

$$v^* = 90 - \delta$$

$$A^* = 360 - RA$$

$$K^* = K + \Delta K$$

In the Azimuth-Elevation System. ϕ latitude of station, θ = sidereal time of exposure

$$d = d^*$$

$$\Delta x = \Delta x^*$$

$$\Delta y = \Delta y^*$$

$$v = 90 - \epsilon$$

$$RA = \theta - t \text{ (t = hour angle)}$$

$$\sin \Delta K = \frac{\cos \phi \sin A}{\cos \delta}$$

$$\sin \delta = \sin \phi \cos v - \cos \phi \sin v \cos A$$

$$\sin t = \frac{\sin v \sin A}{\cos \delta}$$

V. THE ACCURACY OF THE METHOD*

The accuracy of a measuring method is determined by the propagation of the systematic instrumental errors and the random errors of the observations. The basic requirement of any measuring method is that the systematic errors should be sufficiently small so that their influence on the result can be neglected or the systematic errors must be known accurately enough in order to apply corresponding corrections. The systematic errors of an instrument or measuring method may be analyzed by discussing critically the assumptions made in designing the instrument and in deriving the mathematical analysis of the measuring method.

* A detailed study of the accuracy of intersection photogrammetry will be published in a separate report. See reference note on page 34.

The following criteria characterize the systematic errors which we must deal with in the solution of our problem:

1) The photograph produced by the camera should be an exact central projection. To achieve this, the measured plate coordinates must be corrected for distortion. The determination of this distortion is a part of the camera calibration. It is not discussed in this report. Modern measuring lenses are practically distortion free, e. g., the Wild "Aviotar lens", with a field angle of 60° , has for all apertures from 1:4.2 to 1:16 a maximum distortion of less than 4 microns. In general, the distortion curve for a certain lens may be expressed accurately enough as a function of the radial distance from the principal point of the plate and corresponding corrections may be applied directly to the comparator measurements of the plate coordinates.

2) The comparator measurements of the plate coordinates should be affected only by random errors in setting and reading. Hence, it is assumed that the comparator is adjusted and calibrated. The procedure will not be discussed in this report. Especially careful attention must be paid to the perpendicularity of the mechanical axes of the comparator and to the consistency of the scales.

3) The formulas for astronomical and terrestrial refraction must be adequate. The directions of the control points are affected by lateral and vertical refraction. The lateral refraction is neglected. The astronomical refraction is, hence, determined as a function of the elevation angle and the temperature and pressure at the time and location of exposure. Local variations of the refraction coefficient, if known, may be taken into account. Results obtained recently from a wide variety of sources in extensive research on precision trigonometric leveling nets indicate that the refraction anomalies are less than assumed heretofore.

4) The photographic emulsion must represent a plane with sufficient accuracy and no irregular shifts of emulsion must take place.

5) It is assumed that the control points are essentially free of errors. The reduction coefficients of the RA and δ values for date and time of exposure are taken from the American Ephemeris.*

6) The time interval between the orientation exposures of the different stations must be known with sufficient accuracy. (± 0.01 sec = ± 0.15 sec of arc.)

7) The target points must be recorded simultaneously.

* Compare remark on page 50.

8) The relative spatial situation of the camera stations must be known with sufficient accuracy from a local geodetic net.

9) The cameras must not change their interior or exterior orientation during the time interval between the individual orientation exposure and the final target point registration.

The triangulation results of the outlined photogrammetric method will be the less affected by systematic errors the better the above-mentioned conditions are satisfied. If all steps in the arrangement of a photogrammetric measurement are done with the necessary care the results will be affected only by the random errors of the plate measurements. The accuracy of the plate coordinate measurements depends on the image quality. Star images and additional recorded target points are in general measurable within a few microns. The influence of such a random error (m) on the corresponding ray may be expressed with sufficient accuracy by the relation of m to the principal distance (d). Therefore, assuming a mean error of a reading of $\pm 3 \mu$ and $d \sim 300$ mm we have to expect a relative angular deviation

$$\Delta = \frac{3}{300,000} \rho \approx \pm 2''$$

The orientation of the plate will be obtained more accurately, when more stars are carried in the least squares adjustment. However, it should be realized that even with an accurately determined plate orientation, the single spatial ray to any target point will be affected by the entire amount of the error of the target image. Hence, it is sufficient to carry in the least squares adjustment of the orientation so many stars as are necessary to reduce the mean error of the plate orientation to an insignificant fraction of the angular error of the target images.

In general, it may be assumed that with a mean error of $\pm 2-3 \mu$ and a focal length of 300 mm, the individual direction in space will be obtained from a 10-star orientation adjustment with an angular accuracy of 1 to 2 seconds of arc. Hence, the accuracy of the individual photogrammetric camera is 1:100 000 to 1:200 000. The propagation of this error during the triangulation procedure depends on the geometry of the particular configuration.

VI. NUMERICAL EXAMPLES

The first three examples demonstrate the validity of the geometry of the solution expressed by the formulas 10^a , 10^b , and 14 in connection with 19. For each of the three solutions, a rigorous three-star computation was computed. The least squares adjustment of an over-determined solution is shown by three further examples. For all computations, the coordinate system was oriented with respect to the horizon and zenith of the station.

1. Example based on the formula 10a of the projection method.

We use the results of a former calibration as the following approximations:

(length unit is 1 decimeter = 0.1 meter)

$\Delta x_0 = +.00192$	$\sin \kappa_0 = +.00156110$	$A = -.72967230$
$\Delta y_0 = -.00188$	$\cos \kappa_0 = +.99999878$	$B = -.63034344$
$d_0 = +3.01113$	$\sin \nu_0 = +.34099945$	$C = -.26503859$
	$\cos \nu_0 = +.94006349$	$A' = -.59270394$
$\kappa_0 = +0^\circ 5' 22''$		$B' = +.77631620$
$\nu_0 = 19^\circ 56' 16''$	$\sin A_0 = -.62920358$	$C' = -.21455807$
$A_0 = 218^\circ 59' 29''$	$\cos A_0 = -.77724053$	
		$D = +.34099903$
		$E = +.00053233$
		$F = -.94006349$

The standard coordinates of the three stars computed from astronomical data with formula (2) are:

	ξ_c	η_c
3	+.16900891	+.04650153
10	+.15713779	+.38332881
18	+.48127491	+.39613274

Measured plate
coordinates

	l	l'	$l - \Delta x_0$	$l' - \Delta y_0$	s	t
3	+0.21350	-0.57731	+0.21158	-0.57543	-0.51155839	-0.14074964
10	-0.56145	+0.00056	-0.56337	+0.00244	-0.44472948	-1.08486171
18	-0.01032	+0.63807	-0.01224	+0.63995	-1.25730404	-1.03486525

	u	$\xi_0 = \frac{s}{u}$	$\eta_0 = \frac{t}{u}$	$\Delta \xi = \xi_c - \xi_0$	$\Delta \eta = \eta_c - \eta_0$
3	-3.02676182	+.16901178	+.04650172	-.00000287	-.00000019
10	-2.83012124	+.15714149	+.38332694	-.00000370	+.00000187
18	-2.61243757	+.48127621	+.39613014	-.00000130	+.00000260

$$+L = \frac{-\Delta \xi}{\eta_0} \quad +L' = \frac{+\Delta \eta}{\xi_0} \quad \alpha \quad \alpha'$$

3	+.00006172	-.00000112	-17.29801637	+3.76703194
10	+.00000965	+.00001190	- 2.17656068	+4.89034924
18	+.00000328	+.00000540	- 2.71937570	+1.82040370

	β	β'	a	a'	b
3	-3.76092138	-0.62151664	+4.47911206	+1.51750311	+5.59365660
10	+0.40816443	-0.91214623	+0.58111286	+1.74513312	+0.72198824
18	+0.40052769	+0.38794159	+0.60935435	+0.61727737	+0.86367478

	b'	c	c'
3	-1.18962174	-0.75422424	+0.33396677
10	-1.62664740	-0.10813877	-0.32782533
18	-0.57884482	+0.18107826	-0.12553003

$$[a + a'] / 2n = -1.95269464$$

$$[\beta + \beta'] / 2n = -0.68299176$$

$$[a + a'] / 2n = +1.59158214$$

$$[b + b'] / 2n = +0.63070094$$

$$[c + c'] / 2n = -0.13344556$$

$$[L + L'] / 2n = +0.00001514$$

	A (Δv)	B ($\Delta \kappa$)	C ($\Delta \Delta x$)	D ($\Delta \Delta y$)	E (Δd)	L
3	-15.34532173	-3.07792962	+2.88752992	+4.96295566	-0.62077868	-0.00004658
10	-0.22386604	+1.09115619	-1.01046928	+0.09128730	+0.02530679	+0.00000549
18	-0.76668106	+1.08351945	-0.98222779	+0.23297384	+0.31452382	+0.00001186
3	+5.71972658	+0.06147512	-0.07407903	-1.82032268	+0.46741233	+0.00001626
10	+6.84304388	-0.22915447	+0.15355098	-2.25734834	-0.19437977	+0.00000324
18	+3.77309834	+1.07093335	-0.97430477	-1.20954576	+0.00791553	+0.00000974

The normal equations are:*

(Δv)	($\Delta \kappa$)	($\Delta \Delta x$)	($\Delta \Delta y$)
+329.8956076	+48.9810773	-46.3799138	-106.7798210
...	+13.0414762	-12.1376130	-15.8135538
		...+11.3019840	+14.9763025
			...+ 34.5657362
(Δd)	(L)		
+10.6524313	+0.000856387418	=	0
+ 2.3608737	+0.000172898966	=	0
- 2.1992076	-0.000161894590	=	0
- 3.4269437	-0.000276603469	=	0
...+ 0.7412523	+0.000039832489	=	0

* Weighting factors are omitted because this is a unique solution.

Reduction of the normal equations gives:

$$\begin{aligned}\Delta A &= +.00000793 = +1.6'' \\ \Delta v &= +.00000584 = +1.2'' \\ \Delta \kappa &= -.00000646 = -1.3'' \\ \Delta d &= -.00002166 \\ \Delta \Delta x &= -.00000144 \\ \Delta \Delta y &= +.00002157\end{aligned}$$

The final orientation elements are:

(Length unit 1 meter)

$$\begin{aligned}A &= 218^\circ 59' 30.6'' & \sin \kappa &= +.00155480 \\ v &= 19^\circ 56' 17.2'' & \cos \kappa &= +.99999879 \\ \kappa &= +0^\circ 5' 20.7'' & \sin v &= +.34100492 \\ d &= .30111083 & \cos v &= +.94006151 \\ \Delta x &= +.00019186 & \sin A &= -.62920961 \\ \Delta y &= -.00018584 & \cos A &= -.77723565\end{aligned}$$

The check is obtained with formula 10^a

$l - \Delta x$	$l' - \Delta y$	ξ	$\Delta \xi = \xi_c - \xi$ in 8th decimal place	η	$\Delta \eta = \eta_c - \eta$ in 8th decimal place
3 +0.02115814	-0.05754516	+ .16900894	-3	+ .04650148	+5
10 -0.05633686	+0.00024184	+ .15713778	+1	+ .38332876	+5
18 -0.00122386	+0.06399284	+ .48127491	0	+ .39613266	+8

2. Example based on the formula 10^b of the projection method.

We use the results of a former calibration as the following approximations:
(length unit is 1 decimeter = 0.1 meter)

$$\begin{aligned}\Delta x_o &= +.00192 & \sin \kappa_o &= +.79482675 & A &= +.72967232 \\ \Delta y_o &= -.00188 & \cos \kappa_o &= +.60683642 & B &= +.63034344 \\ & & & & C &= +.26503859 \\ d_o &= +3.01113 & \sin \omega_o &= +.21455807 & A' &= +.59270395 \\ \kappa_o &= 52^\circ 38' 20.02'' & \cos \omega_o &= +.97671123 & B' &= -.77631621 \\ \omega_o &= 12^\circ 23' 22.57'' & \sin \alpha_o &= +.27135819 & C' &= +.21455807 \\ \alpha_o &= 15^\circ 44' 42.37'' & \cos \alpha_o &= +.96247843 & D &= -.34099903 \\ & & & & E &= -.00053234 \\ & & & & F &= +.94006349\end{aligned}$$

The standard coordinates of the three stars computed from astronomical data with formula (2) are:

	ξ_c	η_c
3	+ .16900891	+ .04650153
10	+ .15713779	+ .38332881
18	+ .48127491	+ .39613274

Measured plate
coordinates

	l	l'	$l - \Delta x_0$	$l' - \Delta y_0$	s	t
3	+0.21350	-0.57731	+0.21158	-0.57543	+0.51155839	+0.14074964
10	-0.56145	+0.00056	-0.56337	+0.00244	+0.44472948	+1.08486171
18	-0.01032	+0.63807	-0.01224	+0.63995	+1.25730404	+1.03486525
	u	$\xi_0 = \frac{s}{u}$	$\eta_0 = \frac{t}{u}$	$\Delta \xi = \xi_c - \xi_0$	$\Delta \eta = \eta_c - \eta_0$	
3	+3.02676182	+0.16901178	+0.04650172	-0.00000287	-0.00000019	
10	+2.83012124	+0.15714149	+0.38332694	-0.00000370	+0.00000187	
18	+2.61243757	+0.48127621	+0.39613014	-0.00000130	+0.00000260	
	$+L = \frac{\Delta \xi}{1 + \xi_0^2}$	$+L' = \frac{\Delta \eta}{\xi_0 \eta_0}$				
3	-0.00000279	-0.00002418				
10	-0.00000361	+0.00003104				
18	-0.00000106	+0.00001364				
	α	α'	β	β'	a	
3	-0.00491382	+128.5632674	-0.17003235	+19.36545308	-0.20250197	
10	-0.04493296	+19.0340554	+0.15268999	+2.37955149	-0.21738815	
18	+0.06170817	+6.5256801	+0.12882238	-0.97932866	-0.19598763	
	a'	b	b'	c	c'	
3	+3263326183	-0.25289086	-25.58231098	+0.03409870	+7.18181425	
10	+455259704	-0.27008814	-4.24349876	+0.04043356	-0.85521071	
18	+1.55826914	-0.27778514	-1.46124914	-0.05824050	-0.31689090	

$$[\alpha + \alpha'] / 2n = +25.68914405$$

$$[\beta + \beta'] / 2n = +2.47952599$$

$$[a + a'] / 2n = +6.35470838$$

$$[b + b'] / 2n = -5.34797050$$

$$[c + c'] / 2n = +1.00433740$$

$$[L + L'] / 2n = +0.00000217$$

$$\Delta \omega' = 10 \Delta \omega$$

	A ($\Delta \omega$)	B ($\Delta \kappa$)	C ($\Delta \Delta x$)	D ($\Delta \Delta y$)	E (Δd)	L
3	-2.56940578	-2.64955834	-6.55721035	+5.09507964	-0.97023870	+.00000496
10	-2.57340770	-2.32683600	-6.57209653	+5.07788236	-0.96388384	+.00000578
18	-2.56274358	-2.35070361	-6.55069601	+5.07018536	-1.06257790	+.00000323
3	+10.28741234	+10.88592709	+26.27855345	-20.23434048	+6.17747685	+.00002635
10	-.66550886	-0.09997450	-1.80211134	+1.10447174	-1.85954811	-.00002887
18	-1.91634639	-3.45885465	-4.79643924	+3.88672136	-1.32122830	-.00001147

The normal equations are:*

($\Delta \omega$)	($\Delta \kappa$)	($\Delta \Delta x$)	($\Delta \Delta y$)
+129.7380661	+137.5028570	+331.2778473	+255.4947177
	...+148.4372116	+350.9014152	-271.0571917
		...+845.9168847	-652.3570867
			...+503.2065008
(Δd)			
+71.0162461	+.000276370838	= 0	
+79.3147160	+.000295219811	= 0	
+191.6808813	+.000707812766	= 0	
-147.4116336	-.000538643211	= 0	
...+ 46.3642906	+.000217800398	= 0	

* Weighting factors are omitted because this is a unique solution..

Reduction of the normal equations gives:	The final orientation elements are:	(Length unit 1 meter)
$\Delta \alpha = +.00000295 = +0.61''$	$\alpha = 15^\circ 44' 42.98''$	or transformed with formula (11)
$\Delta \omega = +.00000577 = +1.19''$	$\omega = 12^\circ 23' 23.76''$	
$\Delta \kappa = -.00000038 = -0.08''$	$\kappa = 52^\circ 38' 19.94''$	$A = 218^\circ 59' 30.6''$
$\Delta d = -.00002163$	$d = .30111084$	$\nu = 19^\circ 56' 17.2''$
$\Delta \Delta x = -.00000187$	$\Delta x = +.00019181$	$(\kappa) = + 0^\circ 5' 22.7''$
$\Delta \Delta y = +.00002140$	$\Delta y = -.00018586$	
		$\sin \omega = +.21456372$
		$\cos \omega = +.97670999$
		$\sin \alpha = +.27136103$
		$\cos \alpha = +.96247763$
		$\sin \kappa = +.79482651$
		$\cos \kappa = +.60683673$

The check is obtained with formula 10^b.

	$l - \Delta x$	$l' - \Delta y$	ζ	$\frac{\Delta \xi}{\xi_c} = \frac{\xi - \xi_{in}}{\xi_c}$	η	$\frac{\Delta \eta}{\eta_c} = \frac{\eta - \eta_{in}}{\eta_c}$
				8th dec. place		8th dec. place
3	+0.02115819	-0.05754514	+.16900893	-2	+.04650155	-2
10	-0.05633681	+0.00024186	+.15713778	+1	+.38332879	+2
18	-0.00122381	+0.06399286	+.48127489	+2	+.39613276	-2

3. Example based on the formulas (14) and (19) of the analytical method.
We use the results of a former calibration as the following approximations:
(length unit is 1 meter)

$$a_1^0 = +2.22842997$$

$$b_1^0 = +2.57822963$$

$$c_1^0 = +0.28198337$$

$$a_2^0 = -2.74202238$$

$$b_2^0 = +2.09437269$$

$$c_2^0 = +0.22913474$$

$$a_0^0 = +0.00214801$$

$$b_0^0 = -1.20454409$$

And the auxiliaries are:

$$a' = +.06119184$$

$$b' = -1.01063948$$

$$c' = +.26700764$$

$$d' = +.23161468$$

These approximations satisfy the condition equations (19).

$$1) \quad a_0^0 b_0^0 + a_1^0 b_1^0 + a_2^0 b_2^0 = +.000000016$$

$$2) \quad a_0^0 + a_1^0 + a_2^0 - b_0^0 - b_1^0 - b_2^0 = +.000000023$$

The measured plate coordinates l and l' are corrected for distortion and comparator constants.

	l	l'	(1)	(2)	(3)
	(meters)		$a_1^0 l + b_1^0 l' + c_1^0$	$a_2^0 l + b_2^0 l' + c_2^0$	$a_0^0 l + b_0^0 l' + c_0^0$
3	+.021350	-.057731	+.18071658	+.04968233	+1.06958539
10	-.056145	+.000056	+.15701255	+.38320287	+0.99981195
18	-.001032	+.063807	+.44419273	+.36560015	+0.92313944

Computed from formula (2)

	ξ_c	η_c (meters)	$\xi_c^{(4)} \times (3)$	$\eta_c^{(5)} \times (3)$
3	+.16900891	+.04650153	+.18076946	+.04973736
10	+.15713779	+.38332881	+.15710824	+.38325673
18	+.48127491	+.39613274	+.44428385	+.36568576

	$-L=(1)-(4)$ (meters)	$-L'=(2)-(5)$	1η	$1'\eta$
3	-.00005288	-.00005503	+.00099281	-.00268458
10	-.00009569	-.00005386	-.02152200	+.00002147
18	-.00009112	-.00008561	-.00040881	+.02527604

	α	β	γ	δ
3	+.05965167	+.01804449	-.01127907	-.01060254
10	-.00349221	-.05673893	-.00615567	+.01301016
18	-.06454902	+.00286149	+.01499976	-.01343273

$$\begin{aligned}
 [a]/3 &= -.00279652 \\
 [\beta]/3 &= -.01194432 \\
 [\gamma]/3 &= -.00081166 \\
 [\delta]/3 &= -.00367504 \\
 [-L]/3 &= -.00007990 \\
 [L]/3 &= -.01194233 \\
 [L']/3 &= +.00204400 \\
 [L\eta]/3 &= -.00697933 \\
 [L'\eta]/3 &= +.00753764 \\
 [-L']/3 &= -.00006483
 \end{aligned}$$

The observation equations are:*

$$\Delta a_2' = \frac{1}{100} \Delta a_2, \Delta b_2' = \frac{1}{100} \Delta b_2, \Delta a_o' = \frac{1}{100} \Delta a_o, \Delta b_o' = \frac{1}{100} \Delta b_o$$

	A (Δa_2)	B (Δb_2)	C (Δa_o)	D (Δb_o)	-L
3	+.06.244819	+.02.998881	-.01.046741	-.00.692750	+.00002702
10	-.00.069569	-.04.479461	-.00.534401	+.01.668520	-.00001579
18	-.06.175250	+.01.480581	+.01.581142	-.00.975769	-.00001122
3	+.03.329233	-.05.977500	-.00.797214	+.01.022222	+.00000980
10	-.04.420267	-.00.198800	+.01.454267	+.00.751617	+.00001097
18	+.01.091033	+.06.176300	-.00.657052	-.01.773840	-.00002078

*. The decimal point in the coefficient A...D was moved for the convenience of the numerical computations.

The normal equations are: *

$\Delta a'_2$	$\Delta b'_2$	$\Delta a'_0$	$\Delta b'_0$		
+108.9492225	-2.3870512	-26.0627025	-0.2710087	+.000200584298	= 0
...+105.1676876		+2.0138737	-28.2117716	-.000053955515	= 0
		...+7.0634212	-0.2657310	-.000015791011	= 0
			...+8.9723610	+.000021007502	= 0

* Weighting factors can be neglected because this is a unique solution.

The solution of the normal equations is:

$$\Delta a'_0 = -.0000403404 \quad \Delta a_0 = -.00403404$$

$$\Delta b'_0 = -.0000042328 \quad \Delta b_0 = -.00042328$$

or

$$\Delta a'_2 = -.0000115042 \quad \Delta a_2 = -.00115042$$

$$\Delta b'_2 = -.0000001111 \quad \Delta b_2 = -.00001111$$

And with the formulas (44) and (50):

$$\Delta a_1 = -.00106071$$

$$\Delta b_1 = +.00011462$$

$$\Delta c_1 = +.00007172$$

$$\Delta c_2 = +.00007608$$

The plate constants are now:

$$a_1 = +2.22736926$$

$$b_1 = +2.57834425$$

$$c_1 = +0.28205509$$

$$a_2 = -2.74317280$$

$$b_2 = +2.09436158$$

$$c_2 = +0.22921082$$

$$a_0 = -0.00188603$$

$$b_0 = -1.20496737$$

The condition equations reduce to:

$$1) +.00000161$$

$$2) +.00001853$$

If the solution is repeated to eliminate the influence of second order terms, we obtain the following new L and L' terms:

	-L	-L'
3	+.00000075	+ .00000000
10	+.00000089	+ .00000046
18	+.00000132	+ .00000063

and the new absolute terms in the normal equations are:

$$[A(L)] = -.000004877174$$

$$[B(L)] = +.000004016334$$

$$[C(L)] = +.000001078512$$

$$[D(L)] = -.000001090959$$

The corresponding reduction gives:

$$\Delta a_1 = +.00000127$$

$$\Delta b_1 = -.00000399$$

$$\Delta c_1 = -.00000098$$

$$\Delta a_2 = +.00000856$$

$$\Delta b_2 = -.00000312$$

$$\Delta c_2 = -.00000035$$

$$\Delta a_0 = +.00001732$$

$$\Delta b_0 = +.00000313$$

And the final plate constants are:

$$a_1 = +2.22737161$$

$$a_2 = -2.74316815$$

$$a_0 = -0.00187329$$

$$b_1 = +2.57834483$$

$$b_2 = +2.09436279$$

$$b_0 = -1.20496548$$

$$c_1 = +0.28205510$$

$$c_2 = +0.22921080$$

The condition equations reduce now to:

$$1) +.000000025$$

$$2) -.000000069$$

The final check with equations (14) gives:

	ℓ	ℓ'	ξ	$\Delta \xi *$	η	$\Delta \eta *$
3	+0.021350	-.057731	+.16900890	+1	+.04650153	0
10	-.056145	+.000056	+.15713779	0	+.38332882	-1
18	-.001032	+.063807	+.48127492	-1	+.39613273	+1

*The differences $\Delta \xi$ and $\Delta \eta$ are in units of the 8th decimal and show the difference against the originally computed standard coordinates.

The computations of the orientation elements with formula (16) gives:

$$\tan K = +.00155464$$

$$K = +0^\circ 5' 20.67''$$

$$\sin K = +.00155464$$

$$\cos K = +.99999879$$

$$\eta_i = -.13661481$$

$$\xi_i = -.11059631$$

$$\cot (A+K) = +1.23133735$$

$$(A+K) = 219^\circ 4' 51.31''$$

$$\sin (A+K) = -.63041732$$

$$\cos (A+K) = -.77625640$$

$$A = 218^\circ 59' 30.64''$$

$$\sin A = -.62920976$$

$$\cos A = -.77723553$$

$$h' = +2.93250868$$

$$x_c' = -.00019214$$

$$y_c' = +.05311185$$

$$d' = +.88301014$$

$$\sin v = +.34100496$$

$$v = 19^\circ 56' 17.21''$$

$$v/2 = 9^\circ 58' 8.60''$$

$$\tan v/2 = +.17577016$$

$$d = .30111084$$

$$y_o' = +.05292630$$

$$y_o' - y_c' = -.00018555$$

$$\Delta x = +.00019185$$

$$\Delta y = -.00018585$$

The compiled results of the three methods are:

Method	Elements of Orientation						
	d	Δx	Δy	A	v	K	
	m	m	m	o i n	o i n	i n	
Projection Method Formula 10 ^a	.301111	+.000192	-.000186	218 59 30.6	19 56 17.2	+5 20.7	
Projection Method Formula 10 ^b	.301111	+.000192	-.000186	218 59 30.6	19 56 17.2	+5 20.7	
Analytical Method Formulas 14 and 19	.301111	+.000192	-.000186	218 59 30.6	19 56 17.2	+5 20.7	

A comparison of the results shows complete agreement for all three solutions.

4. Example. Least squares solution based on formula 10^a of the projection system.

We use the results of a former calibration as the following approximations.

(length unit is 1 decimeter = 0.1 meter)

$$\begin{aligned} \Delta x_0 &= -.00050 & \sin \kappa_0 &= -.00058178 & A &= -.72953695 \\ \Delta y_0 &= -.00150 & \cos \kappa_0 &= +.99999983 & B &= -.63070287 \\ & & & & C &= -.26455573 \\ d_0 &= 3.01110 & \sin A_0 &= -.63112719 & & \\ & & \cos A_0 &= -.77567936 & A' &= -.59283365 \\ \kappa_0 &= -0^\circ 2' & & & B' &= +.77602439 \\ A_0 &= 219^\circ 8' & \sin v_0 &= +.34106326 & C' &= -.21525430 \\ v_0 &= 19^\circ 56' 30'' & \cos v_0 &= +.94004035 & & \\ & & & & D &= +.34106320 \\ & & & & E &= -.00019842 \\ & & & & F &= -.94004035 \end{aligned}$$

The standard coordinates of the four stars computed from astronomical data with formula (2) are:

	ξ_c	η_c
3	+.16900891	+.04650153
10	+.15713779	+.38332881
17	+.54637688	+.15537271
18	+.48127491	+.39613274

Measured plate coordinates

	l	l'	$l-\Delta x_0$	$l'-\Delta y_0$	s	t
3	+0.21350	-0.57731	+0.21400	-0.57581	-0.5114995	-0.1407234
10	-0.56145	+0.00056	-0.56095	+0.00206	-0.4443139	-1.0846843
17	+0.60320	+0.40158	+0.60370	+0.40308	-1.4714209	-0.4186257
18	-0.01032	+0.63807	-0.00982	+0.63957	-1.2570002	-1.0349314
	u	ξ_0	η_0	$\Delta \xi$ $=\xi_c-\xi_0$	$\Delta \eta$ $=\eta_c-\eta_0$	
3	-3.0269856	+1.6897983	+0.04648962	+0.00002908	+0.00001191	
10	-2.8297416	+1.5701571	+0.38331567	+0.00012208	+0.00001314	
17	-2.6931995	+0.54634679	+0.15543806	+0.00003009	-0.00006535	
18	-2.6124198	+0.48116317	+0.39615815	+0.00011174	-0.00002541	

α	α'	β	β'	a
17.26807537	+3.77905943	-3.77762362	-0.61894165	+4.48162381
-2.17259130	+4.90743739	+0.40617633	-0.90991300	+0.58143343
-6.82466631	+1.30365721	-0.71360684	+0.47757286	+1.50634436
-2.71499269	+1.82481662	+0.39818403	+0.38919941	+0.60932290

	p	p'	$[\alpha + \alpha'] / 2n = -2.14566938$
3	0.0195	0.2690	$[\beta + \beta'] / 2n = -0.54361906$
10	1.1638	0.1753	$[\alpha + \alpha'] / 2n = +1.44843512$
17	0.1417	2.1239	
18	0.8954	1.3955	

$$[L+L^0] / 2n = -0.00017974$$

The normal equations are:

72

Reduction of the normal equations gives:

The final orientation elements are:
(Length unit 1 meter)

$$\begin{aligned}\Delta A &= -.00007911 = -16.3'' & A &= 219^\circ 7' 43.7'' \\ \Delta v &= +.00000627 = +1.3'' & v &= 19^\circ 56' 31.3'' \\ \Delta \kappa &= -.00014390 = -29.7'' & \kappa &= -0^\circ 2' 29.7'' \\ \Delta d &= +.00010869 & d &= .30112087 \\ \Delta \Delta x &= -.00005625 & \Delta x &= -.00005562 \\ \Delta \Delta y &= -.00011844 & \Delta y &= -.00016184 \\ [ppp] &= +.000000008410\end{aligned}$$

The computation of the residuals with formulas (24) and (26):

	ρ	ρ'	v	v'
3	-14.443	+55.218	+21.11	-19.49
10	+37.377	+58.041	+46.55	+14.29
17	-104.804	+7.375	-16.46	-36.57
18	+11.814	-50.596	-41.59	+43.03

The mean error of an observed
1 or 1' value is:

$$[ppp] = 8410.$$

$$[vv] = 8386.$$

$$m = \sqrt{\frac{839}{8-2}} = \pm 6.5\mu$$

The final check is obtained with formula (20) after the residuals are applied.

	$\bar{x} = l + v$	$\bar{y} = l' + v'$	ξ	$\frac{\Delta \xi = \xi_c - \xi}{\xi_c - \xi}$ 8th dec.	η	$\frac{\Delta \eta = \eta_c - \eta}{\eta_c - \eta}$ 8th dec.
3	+.02135211	-.05773295	+.16900889	+2	+.04650156	-3
10	-.05614034	+.00005743	+.15713778	+1	+.38332882	-1
17	+.06031835	+.04015434	+.54637684	+4	+.15537275	-4
18	-.00103616	+.06381130	+.48127490	+1	+.39613276	-2

A certain systematic deviation in the individual parameter of the solution must be expected due to the very unfavorable weight p of point 3. This small number is caused by the procedure which leads to the elimination of ΔA . In the next example this reduction step was not carried out and consequently the normal equation system has six unknowns, but more nearly even weights. A comparison of the results of these least squares adjustments at the end of this paragraph shows the influence on the numerical values.

5. Example. Least squares adjustment based on the formula 10^b of the projection method.

We use the results of a former calibration as the following approximations:

(length unit is 1 decimeter = 0.1 meter)

$\Delta x_0 = -.00050$	$\sin \kappa_0 = +.79465261$	$A = +.72953695$
$\Delta y_0 = -.00150$	$\cos \kappa_0 = +.60706445$	$B = +.63070288$
		$C = +.26455573$
$d_0 = 3.01110$	$\sin \omega_0 = +.21525430$	
	$\cos \omega_0 = +.97655803$	$A' = +.59283366$
$\kappa_0 = 52^\circ 37' 20.84''$		$B' = -.77602439$
$\omega_0 = 12^\circ 25' 49.61''$	$\sin \alpha_0 = +.27090631$	$C' = +.21525430$
$\alpha_0 = 15^\circ 43' 5.54''$	$\cos \alpha_0 = +.96260572$	
		$D = -.34106320$
		$E = +.00019842$
		$F = +.94004035$

The standard coordinates of the four stars computed from astronomical data with formula (2) are:

	ξ_c	η_c
3	+.16900891	+.04650153
10	+.15713779	+.38332881
17	+.54637688	+.15537271
18	+.48127491	+.39613274

Measured plate coordinates

	l	l'	$l - \Delta x_0$	$l' - \Delta y_0$	s	t
3	+0.21350	-0.57731	+0.21400	-0.57581	+0.5114995	+0.1407234
10	-0.56145	+0.00056	-0.56095	+0.00206	+0.4443139	+1.0846843
17	+0.60320	+0.40158	+0.60370	+0.40308	+1.4714209	+0.4186257
18	-0.01032	+0.63807	-0.00982	+0.63957	+1.2570002	+1.0349314

	u	ξ_0	η_0	$+L = \xi_c - \xi_0$	$+L' = \eta_c - \eta_0$
3	+3.0269856	+.16897983	+.04648962	+.00002908	+.00001191
10	+2.8297416	+.15701571	+.38331567	+.00012208	+.00001314
17	+2.6931995	+.54634679	+.15543806	+.00003009	-.00006535
18	+2.6124198	+.48116317	+.39615815	+.00011174	-.00002541

	α	α'	β	β'	γ
3	+1.02855418	+0.00785581	-0.00503228	+1.01046388	-0.17562028
10	+1.02465393	+0.06018658	-0.04590669	+1.14657879	+0.15569375
17	+1.29849481	+0.08492309	+0.03963830	+1.13387202	-0.11092166
18	+1.23151800	+0.19061671	+0.07616699	+1.24402843	+0.15774386
	γ'	a	a'	b	b'
3	+0.10458865	-0.20834898	+0.25637175	-0.26005071	-0.20108770
10	+0.14287064	-0.22287255	+0.27426548	-0.27673524	-0.25570127
17	-0.26092040	-0.23414324	+0.28815364	-0.34006977	-0.23980691
18	-0.18726843	-0.24138824	+0.29708204	-0.34207519	-0.27864918
	c	c'	p	p'	
3	+0.03492183	+0.05667426	9.0061	9.4195	
10	+0.04133050	-0.05126896	7.9205	7.1121	
17	-0.09246708	+0.02567068	5.8661	7.1154	
18	-0.07187094	-0.06015509	5.7050	6.0276	

The normal equations are:

$(\Delta \alpha)$	$(\Delta \omega)$	$(\Delta \kappa)$	$(\Delta \Delta x)$	$(\Delta \Delta y)$
+36.6834722	+3.0979448	-0.4037976	-6.5663873	-10.2379357
...+37.5031466	-1.3546933	+9.1603103	-8.1386151	
	...+1.6279077	-0.3492429	+2.853012	
		...+3.7152807	-0.0603955	
			...+4.2846971	
(Δd)				

-0.6217198	-0.002212233932	= 0
-0.1919527	+0.000487421823	= 0
+0.0152028	-0.000360623451	= 0
+0.0692633	+0.000590375590	= 0
+0.1954602	+0.000506051392	= 0
...+0.1795880	+0.000014184051	= 0
	...+0.000000239045	= 0

Reduction of normal equations gives:

$$\begin{aligned}
 \Delta \alpha &= +0.00001945 = +4.01'' \\
 \Delta \omega &= +0.00000132 = +0.27'' \\
 \Delta \kappa &= +0.00021846 = +45.06'' \\
 \Delta d &= +0.00011235 \\
 \Delta \Delta x &= -0.00011082 \\
 \Delta \Delta y &= -0.00009036 \\
 [ppp] &= +0.000000008325
 \end{aligned}$$

The final orientation elements are:

$$\begin{aligned}
 \alpha &= 15^\circ 43' 9.55'' \\
 \omega &= 12^\circ 25' 49.88'' \\
 \kappa &= 52^\circ 38' 5.90''
 \end{aligned}$$

(Length unit 1 meter)

$$\begin{aligned}
 d &= .30112124 \\
 \Delta x &= -0.00006108 \\
 \Delta y &= -0.00015904
 \end{aligned}$$

Or transformed with formula (11)

$$\begin{aligned}
 A &= 219^\circ 7' 53.65'' \\
 v &= 19^\circ 56' 33.28'' \\
 (\kappa) &= -0^\circ 2' 37.53''
 \end{aligned}$$

The computation of the residuals with formulas (24) and (26):

	ρ	ρ'	v	v'
3	+3.061	+8.554	+14.82	-23.64
10	-13.852	+7.709	+42.71	+15.66
17	+17.271	+4.119	-17.78	-38.54
18	-3.642	-24.652	-43.92	+41.64

The mean error of an observed
l or l' value is:

$$[pp] = 8311.$$

$$[vv] = 8312.$$

$$m = \sqrt{\frac{8311}{8-6}} = \pm 6.5\mu$$

The final check is obtained with formula (32) after the residuals are applied.

	$\bar{x} = l + v$	$\bar{y} = l' + v'$	ξ	$\Delta\xi = \xi_c - \xi$ 8th dec.	η	$\Delta\eta = \eta_c - \eta$ 8th dec.
3	+0.02135148	-0.05773336	+0.16900893	-2	+0.04650157	-4
10	-0.05614073	+0.00005757	+0.15713778	+1	+0.38332882	-1
17	+0.06031822	+0.04015415	+0.54637688	0	+0.15537273	-2
18	-0.00103639	+0.06381116	+0.48127494	-3	+0.39613273	+1

6. Example: Least squares adjustment with four stars based on analytical method formulas (14) and (19).

We use the results of a former calibration as the following approximations:

(Length unit is one meter)

$$a_1^0 = +2.22842997$$

$$b_1^0 = +2.57822963$$

$$c_1^0 = +0.28198337$$

$$a_2^0 = -2.74202238$$

$$b_2^0 = +2.09437269$$

$$c_2^0 = +0.22913474$$

$$a_0^0 = +0.00214801$$

$$b_0^0 = -1.20454409$$

And the auxiliaries are :

$$a' = +0.06119184$$

$$b' = -1.01063948$$

$$c' = +0.26700764$$

$$d' = +0.23161468$$

These satisfy the condition equations (19)

- 1) +.000000016
- 2) +.000000023

The measured plate coordinates l and l' are corrected for distortion and comparator constants.

			(1)	(2)	(3)
	l (meters)	l'	$a_1^0 l + b_1^0 l' + c_1^0$	$a_2^0 l + b_2^0 l' + c_2^0$	$a_0^0 l + b_0^0 l' + 1$
3	+.021350	-.057731	+.18071658	+.04968233	+1.06958539
10	-.056145	+.000056	+.15701255	+.38320287	+0.99981195
17	+.060320	+.040158	+.51993881	+.14784177	+0.95175749
18	-.001032	+.063807	+.44419273	+.36560015	+0.92313944

Computed from formula (2)

	ξ_c	η_c	(4) $\xi_c \times (3)$	(5) $\eta_c \times (3)$
3	+.16900891	+.04650153	+.18076946	+.04973736
10	+.15713779	+.38332881	+.15710824	+.38325673
17	+.54637688	+.15537271	+.52001829	+.14787714
18	+.48127491	+.39613274	+.44428385	+.36568576

	$-L = (1) - (4)$	$-L' = (2) - (5)$	$l \eta$	$l' \eta$
3	-.00005288	-.00005503	+.00099281	-.00268458
10	-.00009569	-.00005386	-.02152200	+.00002147
17	-.00007948	-.00003537	+.00937208	+.00623946
18	-.00009112	-.00008561	-.00040881	+.02527604

	α	β	γ	δ	p	p'
3	+.05965167	+.01804449	-.01127907	-.01060254	.0787	.0823
10	-.00349221	-.05673893	-.00615567	+.01301016	.0792	.0711
17	-.03689417	+.06341911	-.00755037	-.02518991	.0648	.0786
18	-.06454902	+.00286149	+.01499976	-.01343273	.0670	.0707

$$\begin{aligned}
 [\alpha] / 4 &= -.01132093 \\
 [\beta] / 4 &= +.00689654 \\
 [\gamma] / 4 &= -.00249634 \\
 [\delta] / 4 &= -.00905376 \\
 [-L] / 4 &= -.00007979 \\
 [L] / 4 &= +.00612325 \\
 [L'] / 4 &= +.01157250 \\
 [l \eta] / 4 &= -.00289148 \\
 [l' \eta] / 4 &= +.00721310 \\
 [-L'] / 4 &= -.00005747
 \end{aligned}$$

The observation equations are:*

$$\Delta a_2' = \frac{1}{100} \Delta a_2 \quad \Delta b_2' = \frac{1}{100} \Delta b_2 \quad \Delta a_o' = \frac{1}{100} \Delta a_o \quad \Delta b_o' = \frac{1}{100} \Delta b_o$$

	A (Δa_2)	B (Δb_2)	C (Δa_o)	D (Δb_o)	-L
3	+.07.097260	+.01.114795	-.00.878273	-.00.154878	+.00002691
10	+.00.782872	-.06.363547	-.00.365933	+.02.206392	-.00001590
17	-.02.557324	+.05.652257	-.00.505403	-.01.613615	+.00000031
18	-.05.323809	-.00.403505	+.01.749610	-.00.437897	-.00001133
3	+.01.522675	-.06.930350	-.00.388429	+.00.989768	+.00000244
10	-.06.226825	-.01.151650	+.01.863052	+.00.719163	+.00000361
17	+.05.419675	+.02.858550	-.01.226356	+.00.097364	+.00002210
18	-.00.715525	+.05.223450	-.00.248267	-.01.806294	-.00002814

* The decimal point in the coefficients A...D was moved for convenience in the numerical computations.

The normal equations are:

$$\begin{array}{rclclcl}
 (\Delta a_2') & (\Delta b_2') & (\Delta a_o') & (\Delta b_o') & & \\
 +11.6273018 & +0.0301952 & -2.4368121 & +0.4123548 & +.000027579431 & = 0 \\
 & +12.0045505 & -0.4232678 & -2.9733715 & +.000003680403 & = 0 \\
 & & +0.6747305 & +0.0342156 & -.000003973645 & = 0 \\
 & & & +0.9178321 & +.000001339620 & = 0 \\
 & & & & +.000000000181 & = 0
 \end{array}$$

The solution of the normal equation gives:

$$\Delta a_o' = -.0000121886 \quad \Delta a_o = -.00121886$$

$$\Delta b_o' = -.0000062844 \quad \Delta b_o = -.00062844$$

or

$$\Delta a_2' = -.0000046976 \quad \Delta a_2 = -.00046976$$

$$\Delta b_2' = -.0000022811 \quad \Delta b_2 = -.00022811$$

$$ppp = +.0000000000835$$

And with the formulas (44) and (50):

$$\Delta a_1 = -.00043917$$

$$\Delta b_1 = +.00001070$$

$$\Delta c_1 = +.00006731$$

$$\Delta c_2 = +.00006198$$

The plate constants are now:

$$\begin{aligned}a_1 &= +2.22799080 \\b_1 &= +2.57824033 \\c_1 &= +0.28205068 \\a_2 &= -2.74249214 \\b_2 &= +2.09411458 \\c_2 &= +0.22919672 \\a_0 &= +0.00092915 \\b_0 &= -1.20517253\end{aligned}$$

The condition equations reduce to:

$$\begin{aligned}1) &+ .00000090 \\2) &+ .00000146\end{aligned}$$

If the solution is repeated to eliminate the influence of second order terms, we obtain the following new L and L' terms:

	-L	-L'
3	+ .00000269	+ .00000962
10	- .00001445	+ .00000827
17	+ .00001571	+ .00000445
18	- .00000399	- .00002331

and the new absolute terms in the normal equations are:

$$\begin{aligned}[A(L)] &= -.000000005030 \\[B(L)] &= -.000000022879 \\[C(L)] &= +.000000001219 \\[D(L)] &= +.000000006324 \\[L(L)] &= +.000000000083\end{aligned}$$

The corresponding reduction gives:

$$\begin{aligned}\Delta a_1 &= +.00000033 \\ \Delta b_1 &= -.00000019 \\ \Delta c_1 &= +.00000001 \\ \Delta a_2 &= +.00000016 \\ \Delta b_2 &= +.00000007 \\ \Delta c_2 &= -.00000001 \\ \Delta a_0 &= +.00000048 \\ \Delta b_0 &= -.00000055\end{aligned}$$

$$p_{pp} = +.0000000000834$$

And the final plate constants are:

$$\begin{array}{lll} a_1 = +2.22799078 & a_2 = -2.74249198 & a_0 = +0.00092963 \\ b_1 = +2.57824013 & b_2 = +2.09411465 & b_0 = -1.20517308 \\ c_1 = +0.28205069 & c_2 = +0.22919671 & \end{array}$$

The condition equations reduce now to:

$$\begin{array}{l} 1) \quad +.00000003 \\ 2) \quad -.00000010 \end{array}$$

The computation of the residuals gives:

	ρ	ρ'	v	v'
3	+2.709	+9.600	+1.68	-2.32
10	-14.457	+8.254	+4.50	+1.60
17	+15.726	+4.444	-1.54	-3.80
18	-3.978	-22.309	-4.18	+4.21

$$[p\rho\rho] = 834 \quad [vv] = 831$$

Mean error of an observed l or l' value is:

$$m = \sqrt{\frac{834}{8-6}} = \pm 6.5\mu$$

The final check with equation (14) gives:

$l+v$	$l'+v'$	ξ	$\Delta\xi = \xi - \xi_{8th}$ dec.	η	$\Delta\eta = \eta - \eta_{8th}$ dec.
3 +.02135168	-.05773332	+.16900890	+1	+.04650154	-1
10 -.05614050	+.00005760	+.15713779	0	+.38332882	-1
17 +.06031846	+.04015420	+.54637688	0	+.15537272	-1
18 -.00103618	+.06381121	+.48127492	-1	+.39613273	+1

The computation of the orientation elements with formula (16) gives:

$$\tan \kappa = -.00077137$$

$$\kappa = -0^\circ 2' 39.11''$$

$$\sin \kappa = -.00077137$$

$$\cos \kappa = +.99999970$$

$$\eta_i = -.13637460$$

$$\xi_i = -.11095448$$

$$\cot (A+\kappa) = +1.23104244$$

$$(A+\kappa) = 219^\circ 5' 15.49''$$

$$\sin (A + \kappa) = -.63050832$$

$$\cos (A + \kappa) = -.77618249$$

$$\boxed{A = 219^\circ 7' 54.59''}$$

$$\sin A = -.63110685$$

$$\cos A = -.77569591$$

$$h' = +2.93189428$$

$$x_c' = +.00006146$$

$$y_c' = +.05309946$$

$$d' = +.88285555$$

$$\sin v = +.34107642$$

$$\boxed{v = 19^\circ 56' 32.89''}$$

$$v/2 = 9^\circ 58' 16.44''$$

$$\tan v/2 = +.17580936$$

$$d = .30112121$$

$$y_o' = +.05293993$$

$$y_o' - y_c' = -.00015953$$

$$\boxed{\begin{array}{l} \Delta x = -.00006134 \\ \Delta y = -.00015958 \end{array}}$$

The compiled results of the three adjustments are:

Method	Elements of Orientation							Star No.	Residual Corrections	
	d m	Δx m	Δy m	A o i n	v o i n	K i n			v micron	v' micron
Projection Method Formula 10 ^a	.301121	-.000056	-.000162	219 7 43.7	19 56 31.3	-2 29.7		3	+ 2.1	- 1.9
								10	+ 4.7	+ 1.4
								17	- 1.6	- 3.7
								18	- 4.2	- 4.3
Projection Method Formula 10 ^b	.301121	-.000061	-.000159	219 7 53.6	19 56 33.3	-2 37.5		3	+ 1.5	- 2.4
								10	+ 4.3	+ 1.6
								17	- 1.8	- 3.9
								18	- 4.4	+ 4.2
Analytical Method Formulas 14 & 19	.301121	-.000061	-.000160	219 7 54.6	19 56 32.9	-2 39.1		3	+ 1.7	- 2.3
								10	+ 4.5	+ 1.6
								17	- 1.5	- 3.8
								18	- 4.2	+ 4.2

* For results see remark on page 73.

A comparison between the results obtained from the unique 3 stars solution and from the overdetermined 4 stars solution shows an important fact. The square sum of the residuals $[vv] = 83.4$ and consequently the mean error of an individual observed plate coordinate

$$m = \sqrt{\frac{83.4}{8-6}} = \pm 6.5\mu, \text{ or expressed in angular terms with a plate dis-}$$

tance of $d \sim 300$ mm, $m = \pm 4''$. We should now expect that the individual parameters should differ only by a comparably small amount between the two solutions. However, between the unique and overdetermined solutions there are differences of as much as $500''$ for the individual parameters. This shows that the individual parameters are determined with relatively large mean errors. Only their combined effect produces a high precision result in representing an individual ray in space.

Consequently, some of the parameters are able to compensate, at least partly, for errors on the other parameters. This opens the way, e.g., to determine in explicit form the tolerances for a camera design and allows, on the other hand, a shortening of the numerical computations by assuming some of the unknowns as constants. It is obvious that an error in Δx or Δy will be partly compensated for by changes in azimuth or tilt, respectively. The narrower the bundle of rays the more effective is this compensation. By such a measurement the number of unknowns will be reduced to four and therefore the final number of normal equations after the elimination of ΔA will be only three.

VII SUMMARY

A method of high-precision spatial triangulation is described. The relative angular accuracy of the method is better than 1:100000. The mathematical analysis of the plate orientation as well as the triangulation by spatial resection is based on rigorous geometry. A rigorous least squares adjustment determines the most probable plate orientation and delivers the most probable corrections to the measured plate coordinates, thus making it possible to determine the mean error of the orientation elements and of the target point directions. The relative angular mean error of a final spatial direction to any recorded target point is 1 to 2 seconds of arc. The triangulation is based on the principle of photogrammetry by intersection. Hence, each point is determined by combining the measured results of individual camera stations. The mean error of the final coordinates depends on the geometry of the individual triangulation case.

The method is useful for the determination of the spatial position of single points, e. g., for the purpose of calibration of other trajectory measuring methods, as well as for the determination of complete trajectories. The outlined method is free of systematic errors if the conditions discussed in Chapter V are sufficiently satisfied. It must be understood that maximum accuracy with the method can be obtained only if during the time interval between the exposure of the control points for plate orientation and the exposures of the target points, the interior and exterior orientations of the cameras can be considered as unchanged.

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